

Efficient Anderson localization bounds for large multi-particle systems

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Abstract

We study multi-particle interactive quantum disordered systems on a polynomially-growing countable connected graph $(\mathcal{Z}, \mathcal{E})$. The novelty is to give localization bounds uniform in finite or infinite volumes (subgraphs) in \mathcal{Z}^N as well as for the whole of \mathcal{Z}^N . Such bounds are proved here by means of a comprehensive fixed-energy multi-particle multi-scale analysis. Another feature of the paper is that we consider – for the first time in the literature – an infinite-range (although fast-decaying) interaction between particles. For the models under consideration we establish (1) exponential spectral localization, and (2) strong dynamical localization with sub-exponential rate of decay of the eigenfunction correlators.

1 Introduction. The model and results

Until recently, the rigorous Anderson localization theory focused on single-particle models. (In the physical community, notable papers on multi-particle systems with interaction appeared as early as in 2005–2006; see [4, 22].)

Initial rigorous results on multi-particle lattice localization for a finite-range two-body interaction potential were presented in [14–16] and [2, 3]; continuous models have been considered in [6], [13], and later in [23], [27]. In these papers, both Spectral localization (SL) and Dynamical Localization (DL) have been established. A considerable progress was made in [24, 25], with the help of an adapted bootstrap variant of the Multi-Scale Analysis (MSA) developed in the earlier work [21]. The resulting bootstrap multi-particle MSA (MPMSA) was applied in [24] and [25] to multi-particle systems in the lattice and in the Euclidean space, respectively. More recently, a very important step was made in the paper [20] which extended the multi-particle Fractional-Moment Method (MPFMM) from the lattice case [2, 3] to the continuous one, with an infinite-range two-body interaction potential. As usual, (MP)FMM provides, under certain assumptions, exponential decay bounds upon the eigenfunction correlator (EFC), while the bootstrap MPMSA achieves only a sub-exponential decay of the EFC at large distances.

The main motivation for the present work comes from the fact that in all above-mentioned papers the decay bounds on the eigenfunctions and EFCs was proved in the so-called Hausdorff distance (HD) which is actually, a pseudo-distance in the multi-particle configuration space. In the context of the multi-particle Anderson localization, the HD appears explicitly in [2, 3] (as well as in [24, 25]), while in [14–16] it was used implicitly, through the notion of separated cubes. The point is that there are arbitrarily distant loci in the multi-particle

space which might support quantum tunneling between them, and the HD does not reflect this possibility. Another point is that the SL and DL have been proved so far in an infinitely extended physical configuration space, but some tunneling processes could not be ruled out in arbitrarily large, yet bounded subsets thereof. As a result, the existence of efficient multi-particle localization – even for a bounded number of particles $N \geq 3$ – remained an open question. These aspects of the rigorous multi-particle localization theory were analyzed by Aizenman and Warzel in [2, 3], and their analysis of the problem was instrumental for a partial solution given in [7, 9]. The mathematical core of the problem is an eigenvalue concentration (EVC) bound for two distant loci in the multi-particle space, used in different ways in the MPMSA and in MPFMM. In the current paper we employ a probabilistic result from [11] and prove a suitable EVC bound (cf. Theorem 2.2) for a class of sufficiently regular marginal probability distributions of IID external random potentials. We expect such a bound to be extended to a larger family of random potentials. It has to be emphasized that the problem in question appears only for the number of particles $N \geq 3$, and the proof of localization for two-particle systems given in [15] operates with the (symmetrized) norm-distance in the two-particle space. As a result, the two-particle localization holds in finite (but arbitrarily large) regions of the physical configuration space, under mild regularity conditions upon the random potential; see [17].

In the present paper we focus on an interactive N -particle Anderson model, on a countable connected graph $(\mathcal{Z}; \mathcal{E})$ with a polynomially growing size of a ball when the radius increases to infinity. The main method used is a new variant of the MPMSA. The results are summarized as follows.

- We prove uniform localization bounds, in terms of decay of eigenfunctions (EFs) and eigenfunction correlators (EFCs) valid for finite or infinite subgraphs of \mathcal{Z}^N , including the whole \mathcal{Z}^N . Previously published results provided less efficient bounds in finite volumes.
- As in [20], we treat systems with infinite-range interaction potentials. Specifically, we consider a two-body potential decaying at a large distance r as e^{-r^ζ} where $\zeta > 0$. Surprisingly, the SL holds here with an exponential rate (e^{-mr} , $m > 0$) even if $0 < \zeta < 1$. We want to note that an exponential decay of EFs was proved in [8] under the assumption of decay of the interaction with rate e^{-r^ζ} , but only for $\zeta \in (0; 1]$ sufficiently close or equal to 1. Paper [20] established the EFC decay with rate $e^{-\kappa r}$, $\kappa \in (0; 1]$, in the following three cases:
 - The interaction potential decays at an exponential rate e^{-ar} ; in this case the EFCs also decay exponentially fast.
 - The interaction potential decays sub-exponentially, as e^{-r^ζ} with $\zeta \in (0, 1)$; in this case the EFCs also decay sub-exponentially.
 - The interaction potential decays polynomially, as Cr^{-A} , with a sufficiently large $A > 0$; in this case the EFCs also decay sub-exponentially, i.e., at a much faster rate than the interaction.

We present competing bounds for the EFCs. As was said, we also show *exponential* decay of EFs. Together with the results of [8] and [20], this evidences that the decay rate of the EFs and EFCs can be stronger than that of

the interaction potential. The EFC decay bounds are established here in the natural (symmetrized) norm-distance, more efficient than the Hausdorff pseudo-distance.

The rest of the paper is devoted to the proof of Theorem 1.1. In particular, in Section 2 we establish an important ingredient of the proof: eigenvalue concentration bounds (EVCs). The bulk of the work is about the proof of assertion (A): it is carried in Section 3. The main strategy here is the induction on the number of particles N , initially developed in [15, 16]. Each step $N \rightsquigarrow N + 1$, $N = 1, \dots, N^* - 1$, employs the multi-scale analysis of multi-particle Hamiltonians. Unlike Ref. [16], we make use of a more efficient scaling technique, essentially going back to the work [21] and recently adapted in [24] to multi-particle systems.

We need to modify the bootstrap MPMSA strategy from [24]. Specifically, we carry out only two of the four separate scaling analyses which constitute the bootstrap method. This results in a shorter proof of sub-exponential decay of the EFCs. The full-fledged bootstrap MPMSA (cf. [24]), combined with our new eigenvalue concentration estimate (cf. Theorem 2.2), allows us to prove the EFC decay in the symmetrized graph distance. It is worth mentioning that the base of induction ($N = 1$) requires a proof of localization bounds for single-particle systems on graphs with a polynomial growth of the size of a ball with the radius. The required estimates were proved in Ref. [12] following the techniques from [21].

1.1 The multi-particle Hamiltonian

Consider a finite or locally finite, connected non-oriented graph $(\mathcal{Z}, \mathcal{E})$, with the vertex set \mathcal{Z} and the edge set \mathcal{E} . (For brevity, we often refer to \mathcal{Z} only.) We assume that \mathcal{E} does not include cyclic edges $x \leftrightarrow x$ and denote by $d(\cdot, \cdot)$ the graph distance on \mathcal{Z} : $d(x, y)$ equals the length of the shortest path $x \rightsquigarrow y$ over the edges. (By definition, $d(x, x) = 0$.) We assume that graph $(\mathcal{Z}, \mathcal{E})$ belongs to a class $\mathfrak{G}(d, C)$ for some $d, C > 0$, meaning that the size of a ball $\mathcal{B}(x, L) := \{y : d(x, y) \leq L\}$ is polynomially bounded:

$$\sup_{x \in \mathcal{Z}} \#\mathcal{B}(x, L) \leq CL^d, \quad L \geq 1. \quad (1.1)$$

Physically, \mathcal{Z} represents the configuration space of a single quantum particle.

NB. To keep the track of the constants emerging in the course of the presentation, we will use a sub-script indicating their origin; e.g., $C_{\mathcal{Z}}$ will refer to the constant(s) arising in connection with graph $(\mathcal{Z}, \mathcal{E})$ as in the paragraph above. \square

The configuration space of N distinguishable particles is the graph $(\mathcal{Z}^N, \mathcal{E}_N)$. Here \mathcal{Z}^N is the Cartesian power, and the edge set \mathcal{E}_N is defined as follows. Given $\mathbf{x} = (x_1, \dots, x_N), \mathbf{y} = (y_1, \dots, y_N) \in \mathcal{Z}^N$, the edge $\mathbf{x} \leftrightarrow \mathbf{y}$ exists if, for some $j = 1, \dots, N$, there exists an edge $x_j \leftrightarrow y_j$ in \mathcal{E} while for $i \neq j$ we have $x_i = y_i$. We refer to \mathbf{x} and \mathbf{y} as N -particle configurations (briefly, configurations) on \mathcal{Z} and use the same notation $d(\mathbf{x}, \mathbf{y})$ for the graph distance on \mathcal{Z}^N (as before, $d(\mathbf{x}, \mathbf{x}) = 0$). Apart from the distance $d(\cdot, \cdot)$ on \mathcal{Z}^N , it will be convenient to use the max-distance ρ and the symmetrized max-distance ρ_S , defined as follows:

$$\rho(\mathbf{x}, \mathbf{y}) = \max_{1 \leq j \leq N} d(x_j, y_j); \quad \rho_S(\mathbf{x}, \mathbf{y}) = \min_{\pi \in \mathfrak{S}_N} \rho(\mathbf{x}, \pi(\mathbf{y})). \quad (1.2)$$

Here the symmetric group \mathfrak{S}_N acts on \mathcal{Z}^N by permutations of the coordinates.

Next, $\mathcal{B}^{(N)}(\mathbf{x}, L)$ denotes the ball in \mathcal{Z}^N centered at $\mathbf{x} = (x_1, \dots, x_N)$ in metric ρ (sometimes called an N -particle ball):

$$\mathcal{B}^{(N)}(\mathbf{x}, L) := \{\mathbf{y} : \rho(\mathbf{x}, \mathbf{y}) \leq L\} = \bigtimes_{j=1}^N \mathcal{B}(x_j, L). \quad (1.3)$$

It will be often convenient to omit the index N and use the boldface notation: $\mathcal{Z} = \mathcal{Z}^N$, $\mathcal{E} = \mathcal{E}_N$, $\mathcal{B}(\mathbf{x}, L) = \mathcal{B}^{(N)}(\mathbf{x}, L)$, etc. Note that for $\mathcal{Z} \in \mathfrak{G}(d, C)$, one has $\sharp \partial \mathcal{B}^{(N)}(\mathbf{x}, L) \leq C^{2N} L^{Nd}$.

The graph Laplacian $\Delta_{\mathcal{Z}}$ on \mathcal{Z} is given by

$$(\Delta_{\mathcal{Z}} f)(x) = \sum_{\langle x, y \rangle} (f(y) - f(x)) = -n_{\mathcal{Z}}(x)f(x) + \sum_{\langle x, y \rangle} f(y), \quad x \in \mathcal{Z}; \quad (1.4)$$

here $\langle x, y \rangle$ stands for a pair $(x, y) \in \mathcal{Z} \times \mathcal{Z}$ with $d(x, y) = 1$, and $n_{\mathcal{Z}}(x) = \sharp \{y : d(x, y) = 1\}$. Similarly to (1.4), the Laplacian on \mathcal{Z} is defined by

$$\begin{aligned} (\Delta_{\mathcal{Z}} f)(\mathbf{x}) &= \sum_{1 \leq j \leq N} (\Delta_{\mathcal{Z}}^{(j)} f)(\mathbf{x}) \\ &= \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} (f(\mathbf{y}) - f(\mathbf{x})) = -n_{\mathcal{Z}}(\mathbf{x})f(\mathbf{x}) + \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} f(\mathbf{y}), \quad \mathbf{x} \in \mathcal{Z}. \end{aligned} \quad (1.5)$$

Here $\Delta_{\mathcal{Z}}^{(j)}$ denotes the Laplacian acting on the j th component of \mathbf{x} , $\langle \mathbf{x}, \mathbf{y} \rangle$ stands for a pair $(\mathbf{x}, \mathbf{y}) \in \mathcal{Z} \times \mathcal{Z}$ with $d(\mathbf{x}, \mathbf{y}) = 1$ and $n_{\mathcal{Z}}(\mathbf{x}) = \sharp \{y : d(\mathbf{x}, y) = 1\}$.

Given $\mathcal{V} \subset \mathcal{Z}$, we write $\langle x, y \rangle \in \mathcal{V}$ meaning that $x, y \in \mathcal{V}$ and $d(x, y) = 1$. Likewise, for $\mathcal{V} \subseteq \mathcal{Z}$ the notation $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathcal{V}$ means $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $d(\mathbf{x}, \mathbf{y}) = 1$. With this agreement, the Laplacians $\Delta_{\mathcal{V}}$ and $\Delta_{\mathcal{V}}$ (with Dirichlets boundary condition) are introduced as follows:

$$\Delta_{\mathcal{V}} = \mathbf{1}_{\mathcal{V}} \Delta_{\mathcal{Z}} \mathbf{1}_{\mathcal{V}} \quad (1.6)$$

and

$$\Delta_{\mathcal{V}} = \mathbf{1}_{\mathcal{Z}} \Delta_{\mathcal{Z}} \mathbf{1}_{\mathcal{Z}}. \quad (1.7)$$

The N -particle Hamiltonian $\mathbf{H}_{\mathcal{V}}^{(N)} = \mathbf{H}_{\mathcal{V}}^{(N)}(\omega)$ in ‘volume’ $\mathcal{V} \subseteq \mathcal{Z}$ acts as

$$\begin{aligned} (\mathbf{H}_{\mathcal{V}}^{(N)} f)(\mathbf{x}) &= (-\Delta_{\mathcal{V}} f)(\mathbf{x}) + g \sum_{1 \leq j \leq N} V(x_j; \omega) f(\mathbf{x}) \\ &\quad + \sum_{1 \leq i < j \leq N} U(d(x_i, x_j)) f(\mathbf{x}), \quad \mathbf{x} = (x_1, \dots, x_N) \in \mathcal{V}. \end{aligned} \quad (1.8)$$

Here V represents a random external field and U a two-body interaction; see below. The constant $g \in \mathbb{R}$ is referred to as a coupling amplitude. Under the imposed conditions, with probability one, the operator $\mathbf{H}_{\mathcal{V}}^{(N)}(\omega)$ is bounded and self-adjoint in $\ell_2(\mathcal{V})$.

1.2 The assumptions and results

Our goal is to prove that, under certain conditions on $V(\cdot; \omega)$ (the external potential) and $U(\cdot)$ (the two-body interaction potential) and for sufficiently

large values of the disorder amplitude $|g|$, the (random) eigenvectors of $\mathbf{H}_{\mathcal{V}}^{(N)}$ in $\ell_2(\mathcal{V})$ feature strong decay properties, stated in appropriate terms. We stress that we establish the threshold for $|g|$ (and other bounds involved) uniformly in \mathcal{V} for a bounded range of values of N . Formal statements are given in Theorem 1.1 below.

The condition upon V is:

(V): *The random field $(x, \omega) \mapsto V(x; \omega) \in \mathbb{R}$ is IID, with a marginal probability distribution supported by a bounded interval and admitting a smooth probability density p_V satisfying the following conditions: $\forall t \in \text{supp } p_V$,*

$$0 < \underline{p} \leq p_V(t) \leq \bar{p} < \infty; \quad |p'_V(t)| \leq R < \infty. \quad (1.9)$$

The probability distribution generated by the random variables $V(x)$ is denoted by \mathbb{P} and the expectation by \mathbb{E} .

We assume the following condition upon U .

(U): $\exists \zeta > 0$ and $C = C_U > 0$ such that

$$|U(r)| \leq C e^{-r^\zeta}, \quad r = 1, 2, \dots \quad (1.10)$$

When $\zeta \in (0, 1)$, it makes sense to refer to a sub-exponential decay of U , and with $\zeta = 1$ we have the case of an exponential decay.

Let \mathcal{B}_1 denote the set of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{C}$ with $\|f\|_\infty \leq 1$. The results of this paper are presented in Theorem 1.1.

Theorem 1.1. *Assume Conditions (V) and (U) and fix an integer $N^* \geq 2$. $\exists \kappa^* = \kappa^*(\zeta, N^*) \in (0, \zeta]$ with the following property. $\forall \kappa \in (0, \kappa^*]$ and $m > 0$, there is a value $g_0 = g_0(N^*, m, \kappa) > 0$ such that $\forall N = 1, \dots, N^*$ and $|g| \in (g_0, +\infty)$:*

(A) \exists a constant $C = C_{\text{EFC}} > 0$ such that \forall set $\mathcal{V} \subseteq \mathcal{Z}$ (finite or infinite) and $\forall \mathbf{x}, \mathbf{y} \in \mathcal{V}$,

$$\mathbb{E} \left[\sup_{f \in \mathcal{B}_1} |\langle \mathbf{1}_{\mathcal{V}} | f(\mathbf{H}_{\mathcal{V}}^{(N)}) | \mathbf{1}_{\mathcal{X}} \rangle| \right] \leq C e^{-m(\rho_{\mathcal{S}}(\mathbf{x}, \mathbf{y}))^\kappa}. \quad (1.11)$$

(B) *With probability one, $\mathbf{H}_{\mathcal{Z}}^{(N)}(\omega)$ has pure point spectrum. and all its eigenfunctions $\Psi_j(x; \omega)$ decay exponentially fast: there exists a nonrandom number $m = m_N > 0$ such that $\forall \Psi_j \exists$ a constant $C_j = C_j(\omega)$ and a site $\hat{\mathbf{x}}_j = \hat{\mathbf{x}}_j(\omega) \in \mathcal{Z}$ (a localization center) such that*

$$|\Psi_j(\mathbf{x}, \omega)| \leq C_j(\omega) e^{-m \rho_{\mathcal{S}}(\mathbf{x}, \hat{\mathbf{x}}_j)}, \quad \mathbf{x} \in \mathcal{V}. \quad (1.12)$$

The quantity in the LHS of (1.11) is called the EF correlator (EFC), between \mathbf{x} and \mathbf{y} , for Hamiltonian $\mathbf{H}_{\mathcal{V}}^{(N)}$. Compared with [3, 16, 24], Eqns (1.12)–(1.11) show the decay in a more suitable form involving metric $\rho_{\mathcal{S}}$ rather than the Hausdorff distance.

The rest of the paper is devoted to the proof of Theorem 1.1. In particular, in Section 2 we establish an important ingredient of the proof: eigenvalue concentration bounds. The bulk of the work is about the proof of assertion

(A): it is carried in Section 3. The main strategy here is the induction on the number of particles N , initially developed in [15, 16]. Each step $N \rightsquigarrow N + 1$, $N = 1, \dots, N^* - 1$, employs the multi-scale analysis of multi-particle Hamiltonians. Unlike Ref. [16], we make use of a more efficient scaling technique, essentially going back to the work by Germinet and Klein [21] and recently adapted by Klein and Nguyen [24] to multi-particle systems. However, we do not follow closely the bootstrap (MP)MSA strategy from [24]. Specifically, we carry out only two of the four separate scaling analyses which constitute the bootstrap method. This results in a shorter proof of sub-exponential decay of the EFC, with rate e^{-L^κ} for some $\kappa > 0$. The full-fledged bootstrap MPMSA (cf. [24]), combined with our new eigenvalue concentration estimate (cf. Theorem 2.2), would allow one to prove the EFC decay in the symmetrized graph-distance (and not only in the Hausdorff distance), with rate e^{-L^κ} for $\kappa \in (0, 1)$ arbitrarily close to 1, while starting from fairly weak assumptions upon the localization properties in the balls of radius L_0 .

It is worth mentioning that the base of induction ($N = 1$) requires a proof of localization bounds for single-particle systems on graphs of polynomial growth of balls, and we cannot simply refer to [24] where, formally speaking, only the lattice systems (on \mathbb{Z}^d , $d \geq 1$) were studied. The required estimates for the 1-particle Anderson models on graphs were proved in Ref. [12], where it was emphasized that the main scaling technique is due to Germinet and Klein [21].

The proof of assertion **(B)** is contained in Section 4 (it makes use of a number of facts established in other sections). This proof is based on the (modified) version of the MPMSA presented in [17]. In particular, the case of the two-body potential U satisfying (U) is treated as a small perturbation of a finite range interaction. Some technical proofs are presented in the Appendix. Others repeat arguments published elsewhere (sometimes with minor changes) and are omitted.

From here on we fix a positive integer $N^* \geq 2$ and consider $N = 1, \dots, N^*$ without stressing it every time again. The dependence of various quantities upon the upper-bound value N^* is not emphasized but of course is crucial throughout the whole construction. Conditions (V) and (U) are also globally assumed, although a number of intermediate assertions (particularly in Section 2) require more liberal restrictions upon V .

2 Eigenvalue concentration bounds

2.1 The resolvent inequalities. Singular and resonant sets

The main tool in the proof of Theorem 1.1 are properties of decay of the Green functions (GFs) of Hamiltonian $\mathbf{H}_{\mathcal{V}}^{(N)}$:

$$(\mathbf{x}, \mathbf{y}) \in \mathcal{V} \times \mathcal{V} \mapsto G_{\mathcal{V}}(\mathbf{x}, \mathbf{y}) \left(= G_{\mathcal{V}}^{(N)}(\mathbf{x}, \mathbf{y}; E; \omega) \right),$$

for a finite set $\mathcal{V} \subset \mathcal{Z}$ and the value $E \notin \Sigma \left(\mathbf{H}_{\mathcal{V}}^{(N)} \right)$. As usual, $G_{\mathcal{V}}(\mathbf{x}, \mathbf{y})$ denotes the matrix entry of the resolvent $\mathbf{G}_{\mathcal{V}}(E) = \mathbf{G}_{\mathcal{V}}^{(N)}(E; \omega)$ in the delta-basis :

$$G_{\mathcal{V}}(\mathbf{x}, \mathbf{y}) = \left\langle \mathbf{1}_{\mathbf{x}}, \left(\mathbf{H}_{\mathcal{V}}^{(N)} - E\mathbf{I} \right)^{-1} \mathbf{1}_{\mathbf{y}} \right\rangle.$$

The base for the argument is the Geometric resolvent inequality (GRI) for the GFs: \forall subset $\mathcal{W} \subset \mathcal{V}$ and configurations $\mathbf{x} \in \mathcal{W}$, $\mathbf{y} \in \mathcal{V} \setminus \mathcal{W}$,

$$|G_{\mathcal{V}}(\mathbf{x}, \mathbf{y})| \leq \sum_{\langle \mathbf{u}, \mathbf{v} \rangle \in \partial_{\mathcal{V}} \mathcal{W}} |G_{\mathcal{W}}(\mathbf{x}, \mathbf{u})| |G_{\mathcal{V}}(\mathbf{v}, \mathbf{y})| \quad (2.1)$$

Here $\partial_{\mathcal{V}} \mathcal{W}$ stands for the edge-boundary of \mathcal{W} in \mathcal{V} :

$$\partial_{\mathcal{V}} \mathcal{W} = \{(\mathbf{u}, \mathbf{v}) : \mathbf{u} \in \mathcal{W}, \mathbf{v} \in \mathcal{V} \setminus \mathcal{W}, \rho(\mathbf{u}, \mathbf{v}) = 1\}.$$

The distance dist below refers to the standard metric on the line \mathbb{R} . The inner boundary $\partial^- \mathcal{V}$ is determined by

$$\partial^- \mathcal{V} = \{\mathbf{u} \in \mathcal{V} : \rho(\mathbf{u}, \mathcal{Z} \setminus \mathcal{V}) = 1\}.$$

Definition 2.1. Given $E \in \mathbb{R}$, $\beta, \delta \in (0, 1)$ and $m > 0$ an N -particle ball $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{x}, L) \subset \mathcal{Z}$ is called

- (E, β) -resonant $((E, \beta)$ -R, in short), if

$$\text{dist} \left(\Sigma \left(\mathbf{H}_{\mathcal{B}}^{(N)} \right), E \right) < 2e^{-L^\beta} \quad (2.2)$$

and (E, β) -nonresonant $((E, \beta)$ -NR), otherwise;

- (E, δ, m) -nonsingular $((E, \delta, m)$ -NS), if for all configurations $\mathbf{y} \in \partial^- \mathcal{B}$

$$\left| G_{\mathcal{B}}^{(N)}(\mathbf{x}, \mathbf{y}; E) \right| \leq (C_{\mathcal{Z}}^{2N} L^{Nd})^{-1} e^{-mL^\delta}, \quad (2.3)$$

and (E, δ, m) -singular $((E, \delta, m)$ -S), otherwise.

Typically, properties (E, δ, m) -NS and (E, δ, m) -S will be used with $m = m_N$ where m_N varies in a certain specified manner (see (3.1) and (4.2)).

2.2 One- and two-volume EVC bounds

We start with a one-volume EVC bound that is an analog of the well-known Wegner-type estimate:

Theorem 2.1. Fix $\beta \in (0, 1)$ There exists a constant $C = C_{\beta, V}^{(1)}$ such that $\forall E \in \mathbb{R}$, $1 \leq N \leq N^*$, $\mathbf{x} \in \mathcal{Z}$ and integer $L > 1$,

$$\sup_{E \in \mathbb{R}} \mathbb{P} \left\{ \text{ball } \mathcal{B}^{(N)}(\mathbf{x}, L) \text{ is } (E, \beta)\text{-R} \right\} \leq C e^{-L^{\beta/2}}. \quad (2.4)$$

The proof of Theorem 2.1 is omitted: it repeats the one given in [17], Theorem 3.4.1 (Eqn (3.41)). Theorem 2.1 is used in the proof of Theorems 3.5 and 4.5.

A (new) two-volume EVC is the subject of Theorem 2.2 below. It is instructive to compare it with Theorem 3.4.2 (Eqn (3.44)), Corollary 3.1 (Eqns (3.47)–(3.48)) and Theorem 3.5.2 (Eqn (3.58)) in [17].

Given an integer $R \geq 0$, we will say that two balls $\mathcal{B}^{(N)}(\mathbf{x}, L)$, $\mathcal{B}^{(N)}(\mathbf{y}, L)$ are R -distant if

$$\rho_S(\mathbf{x}, \mathbf{y}) \geq R. \quad (2.5)$$

Theorem 2.2. *There is a constant $C = C_V^{(2)}$ such that for all $s > 0$, integer $L > 1$ and any pair of $3NL$ -distant balls $\mathcal{B}^{(N)}(\mathbf{x}, L)$, $\mathcal{B}^{(N)}(\mathbf{y}, L)$, the spectra $\Sigma_{\mathbf{x}} := \Sigma(\mathbf{H}_{\mathcal{B}(\mathbf{x}, L)}^{(N)})$, $\Sigma_{\mathbf{y}} := \Sigma(\mathbf{H}_{\mathcal{B}(\mathbf{y}, L)}^{(N)})$ obey*

$$\mathbb{P}\{\text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s\} \leq C L^{(2N+1)d} s^{2/3}. \quad (2.6)$$

Proof. The proof of Theorem 2.2 will be obtained by collecting the assertions of Theorem 2.3 and Lemmas 2.4 and 2.5.

Given a random field $V(x; \omega)$, $x \in \mathcal{Z}$, and a finite subset $\mathcal{Q} \subset \mathcal{Z}$, consider the sample mean and the fluctuations of V relative to \mathcal{Q} :

$$\xi_{\mathcal{Q}}(\omega) := (\#\mathcal{Q})^{-1} \sum_{x \in \mathcal{Q}} V(x; \omega), \quad \eta_x(\omega) = \eta_{x, \mathcal{Q}}(\omega) = V(x; \omega) - \xi_{\mathcal{Q}}(\omega),$$

and the sigma-algebra $\mathfrak{F}_{\mathcal{Q}}$ generated by the fluctuations $\{\eta_x, x \in \mathcal{Q}\}$ and by $\{V(y; \omega), y \notin \mathcal{Q}\}$.

We use the following property reflecting regularity of the conditional mean:

(RCM): *There exist constants $C', C'', A', A'', b', b'' \in (0, +\infty)$ such that for any finite subset $\mathcal{Q} \subset \mathcal{Z}$, the conditional distribution function $F_{\xi}(\cdot | \mathfrak{F}_{\mathcal{Q}})$ of the sample mean $\xi_{\mathcal{Q}}$ satisfies for all $s \in (0, 1)$*

$$\mathbb{P}\left\{\sup_{t \in \mathbb{R}} |F_{\xi}(t + s | \mathfrak{F}_{\mathcal{Q}}) - F_{\xi}(t | \mathfrak{F}_{\mathcal{Q}})| \geq C' (\#\mathcal{Q})^{A'} s^{b'}\right\} \leq C'' (\#\mathcal{Q})^{A''} s^{b''}. \quad (2.7)$$

Condition (RCM) is fulfilled for an IID Gaussian field, e.g., with zero mean and a unit variance; in this case the sample mean is independent of the fluctuations η_{\bullet} and has a normal distribution with variance $\sigma^2 = (\#\mathcal{Q})^{-1}$. An elementary argument (cf. [11]) shows that (RCM) also holds for an IID random field with a uniform marginal distribution. Moreover, using standard approximation techniques, one can prove the following result:

Theorem 2.3 (Cf. Theorem 6 in [11]). *If a random field $x \in \mathcal{Z} \mapsto V(x; \omega) \in \mathbb{R}$ obeys (V), then it satisfies property (RCM) with*

$$C' = 1, A' = 1, b' = 2/3, C'' = (4R\bar{p})^2, A'' = 0, b'' = 2/3.$$

Before we move further, let us introduce some notation. In Eqn (2.8) we define the support $\Pi \mathbf{x}$ of the configuration $\mathbf{x} = (x_1, \dots, x_N) \in \mathcal{Z}^N$, the support $\Pi \mathcal{B}(\mathbf{x}, L)$ of the ball $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{x}, L)$, and – given a subset $\mathcal{J} \subset \{1, N\}$ – the partial supports $\Pi_{\mathcal{J}} \mathbf{x}$ and $\Pi_{\mathcal{J}} \mathcal{B}$:

$$\begin{aligned} \Pi \mathbf{x} &= \bigcup_{1 \leq i \leq N} \{x_i\} \subset \mathcal{Z}, \quad \Pi \mathcal{B} = \bigcup_{1 \leq j \leq N} \mathcal{B}(x_j, L) \subset \mathcal{Z}, \\ \Pi_{\mathcal{J}} \mathbf{x} &= \bigcup_{j \in \mathcal{J}} \{x_j\} \subset \mathcal{Z}, \quad \Pi_{\mathcal{J}} \mathcal{B} = \bigcup_{j \in \mathcal{J}} \mathcal{B}(x_j, L) \subset \mathcal{Z}, \end{aligned} \quad (2.8)$$

with $\Pi_{\emptyset} \mathbf{x} = \emptyset$ (for $\mathcal{J} = \emptyset$).

Definition 2.2. *A ball $\mathcal{B}^{(N)}(\mathbf{x}, L)$ is called weakly separated from $\mathcal{B}^{(N)}(\mathbf{y}, L)$ if there exists a single-particle ball $\mathcal{B} \subset \mathcal{Z}$, of diameter $\text{diam} \mathcal{B} \leq 2NL$, and subsets $\mathcal{J}_1, \mathcal{J}_2 \subset \{1, \dots, N\}$ such that $\#\mathcal{J}_1 > \#\mathcal{J}_2$ (possibly, with $\mathcal{J}_2 = \emptyset$) and*

$$\begin{aligned} \Pi_{\mathcal{J}_1} \mathcal{B}^{(N)}(\mathbf{x}, L) \cup \Pi_{\mathcal{J}_2} \mathcal{B}^{(N)}(\mathbf{y}, L) &\subset \mathcal{B}, \\ \Pi_{\mathcal{J}_1^c} \mathcal{B}^{(N)}(\mathbf{y}, L) \cup \Pi_{\mathcal{J}_2^c} \mathcal{B}^{(N)}(\mathbf{x}, L) &\subset \mathcal{Z} \setminus \mathcal{B}. \end{aligned} \quad (2.9)$$

A pair of balls $\mathcal{B}^{(N)}(\mathbf{x}, L)$, $\mathcal{B}^{(N)}(\mathbf{y}, L)$ is called weakly separated if at least one of the balls is weakly separated from the other.

To stress the role of the ball \mathcal{B} , we will say, where appropriate, that $\mathcal{B}^{(N)}(\mathbf{x}, L)$ and $\mathcal{B}^{(N)}(\mathbf{y}, L)$ are weakly \mathcal{B} -separated.

Lemma 2.4. (Cf. Lemma 2.3 in [7]) Any pair of $3NL$ -distant balls $\mathcal{B}^{(N)}(\mathbf{x}, L)$, $\mathcal{B}^{(N)}(\mathbf{y}, L)$ is weakly separated.

The proof of Lemma 2.4 repeats that of Lemma 2.3 in [7] and is omitted.

Lemma 2.5. Let $(x, \omega) \rightarrow V(x; \omega)$ be a random field satisfying the condition (RCM). Assume that the balls $\mathcal{B}^{(N)}(\mathbf{x}, L)$, $\mathcal{B}^{(N)}(\mathbf{y}, L)$ are weakly separated. Then for any $s > 0$ the following bound holds for the spectra $\Sigma_{\mathbf{x}} := \Sigma(\mathbf{H}_{\mathcal{B}(\mathbf{x}, L)}^{(N)})$ and $\Sigma_{\mathbf{y}} := \Sigma(\mathbf{H}_{\mathcal{B}(\mathbf{y}, L)}^{(N)})$:

$$\begin{aligned} \mathbb{P} \{ \text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s \} \\ \leq (\#\mathcal{B}^{(N)}(\mathbf{x}, L)) (\#\mathcal{B}^{(N)}(\mathbf{y}, L)) C' L^{A'} (2s)^{b'} + C'' L^{A''} (2s)^{b''} \end{aligned} \quad (2.10)$$

where $A', A'', C', C''b', b'' \in (0, \infty)$ are as in (2.7).

Proof. Let \mathcal{B} be a ball satisfying the conditions (2.9) for some $\mathcal{J}_1, \mathcal{J}_2 \subset \{1, \dots, N\}$ with $\#\mathcal{J}_1 = n_1 > n_2 = \#\mathcal{J}_2$. Introduce the sample mean $\xi = \xi_{\mathcal{B}}$ of V over \mathcal{B} and the respective fluctuations $\{\eta_x := V(x; \omega) - \xi_{\mathcal{B}}(\omega), x \in \mathcal{B}\}$.

Operators $\mathbf{H}_{\mathcal{B}(\mathbf{x}, L)}^{(N)}(\omega)$, $\mathbf{H}_{\mathcal{B}(\mathbf{y}, L)}^{(N)}(\omega)$ read as follows:

$$\mathbf{H}_{\mathcal{B}(\mathbf{x}, L)}^{(N)}(\omega) = n_1 \xi(\omega) \mathbf{I} + \mathbf{A}'(\omega), \quad \mathbf{H}_{\mathcal{B}(\mathbf{y}, L)}^{(N)}(\omega) = n_2 \xi(\omega) \mathbf{I} + \mathbf{A}''(\omega) \quad (2.11)$$

where operators $\mathbf{A}'(\omega)$ and $\mathbf{A}''(\omega)$ are $\mathfrak{F}_{\mathcal{B}}$ -measurable. Let

$$\Sigma_{\mathbf{x}} = \{\lambda_1, \dots, \lambda_{K'}\} \quad \text{and} \quad \Sigma_{\mathbf{y}} = \{\mu_1, \dots, \mu_{K''}\},$$

where $K' = \#\mathcal{B}(\mathbf{x}, L)$ and $K'' = \#\mathcal{B}(\mathbf{y}, L)$.

Owing to (2.11), we have $\lambda_j(\omega) = n_1 \xi(\omega) + \lambda_j^{(0)}(\omega)$, $\mu_j(\omega) = n_2 \xi(\omega) + \mu_j^{(0)}(\omega)$, where the random variables $\lambda_j^{(0)}(\omega)$ and $\mu_j^{(0)}(\omega)$ are $\mathfrak{F}_{\mathcal{B}}$ -measurable. Therefore,

$$\lambda_i(\omega) - \mu_j(\omega) = (n_1 - n_2) \xi(\omega) + (\lambda_j^{(0)}(\omega) - \mu_j^{(0)}(\omega)),$$

with $n_1 - n_2 \geq 1$, owing to our assumption. Further, we can write

$$\mathbb{P} \{ \text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s \} \leq \sum_{1 \leq i \leq K'} \sum_{1 \leq j \leq K''} \mathbb{E} \left[\mathbb{P} \{ |\lambda_i - \mu_j| \leq s \mid \mathfrak{F}_{\mathcal{B}} \} \right].$$

Note that for all i and j we have

$$\begin{aligned} \mathbb{P} \{ |\lambda_i - \mu_j| \leq s \mid \mathfrak{F}_{\mathcal{B}} \} &= \mathbb{P} \left\{ |(n_1 - n_2) \xi + \lambda_i^{(0)} - \mu_j^{(0)}| \leq s \mid \mathfrak{F}_{\mathcal{B}} \right\} \\ &\leq \nu_L (2|n_1 - n_2|^{-1} s \mid \mathfrak{F}_{\mathcal{B}}) \leq \nu_L (2s \mid \mathfrak{F}_{\mathcal{B}}). \end{aligned}$$

Set

$$\mathcal{D}_L = \left\{ \omega : \sup_{t \in \mathbb{R}} |F_{\xi}(t + s \mid \mathfrak{F}_{\mathcal{B}}) - F_{\xi}(t \mid \mathfrak{F}_{\mathcal{B}})| \geq C' L^{A'} s^{b'} \right\}.$$

By (RCM), $\mathbb{P}\{\mathcal{D}_L\} \leq C'' L^{A''} s^{b''}$. Therefore, denoting $\mathcal{D}_L^c = \Omega \setminus \mathcal{D}_L$,

$$\begin{aligned} \mathbb{P}\{\text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s\} &\leq \mathbb{E}\left[\mathbf{1}_{\mathcal{D}_L^c} \mathbb{P}\{\text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s \mid \mathfrak{F}_{\mathcal{B}}\}\right] + \mathbb{P}\{\mathcal{D}_L\} \\ &\leq (\sharp \mathcal{B}(\mathbf{x}, L)) \cdot (\sharp \mathcal{B}(\mathbf{y}, L)) C' L^{A'} s^{b'} + C'' L^{A''} s^{b''}, \end{aligned} \quad (2.12)$$

as claimed in (2.10). This finishes the proof of Lemma 2.5. \square

By Theorem 2.3, property (RCM) is fulfilled with $b' = b'' = 2/3$, $A' = 1$, $A'' = 0$. Hence, the RHS in (2.12) is bounded by $\overline{C} L^{Nd} s^{2/3}$. This completes the proof of Theorem 2.2.

Theorem 2.2 is essential in the proof of Theorem 3.8. Namely, it allows us to infer from the fixed-energy decay bounds (which are simpler to establish) their energy-interval counterparts, required for the proof of spectral and dynamical localization, without an additional scaling analysis employed in the bootstrap multi-scale approach (cf. [24], [21]).

2.3 Weakly interactive balls

Definition 2.3. An N -particle ball $\mathcal{B}^{(N)}(\mathbf{u}, L)$, with $N \geq 2$, centered at $\mathbf{u} = (u_1, \dots, u_n)$, $u_i \in \mathcal{Z}$, is called *weakly interactive (WI)* if

$$\text{diam}(\Pi \mathbf{u}) := \max_{1 \leq i < j \leq N} d(u_i, u_j) > 3NL, \quad (2.13)$$

and *strongly interactive (SI)*, otherwise.

The meaning of Definition 2.3 is that a particle system in a WI ball can be decomposed into distant subsystems that interact “weakly” with each other, whereas for an SI ball such a decomposition is not possible. See Lemma 2.6.

Lemma 2.6. For any WI ball $\mathcal{B}^{(N)}(\mathbf{u}, L)$ there exists a decomposition $\{1, \dots, N\} = \mathcal{J} \cup \mathcal{J}^c$, with $\mathcal{J}^c := \{1, \dots, N\} \setminus \mathcal{J}$, such that,

$$d\left(\Pi_{\mathcal{J}} \mathcal{B}^{(N)}(\mathbf{u}, L), \Pi_{\mathcal{J}^c} \mathcal{B}^{(N)}(\mathbf{u}, L)\right) > L. \quad (2.14)$$

Proof. Suppose that $\text{diam}(\Pi \mathbf{u}) > 3NL$; we want to show that the projection $\Pi \mathcal{B}(\mathbf{u}, 3L/2)$ is a disconnected subset of \mathcal{Z} .

Assume otherwise; then every partial projection $\Pi_{\mathcal{J}} \mathbf{u}$, $\emptyset \subset \mathcal{J} \subset \{1, \dots, N\}$, is at distance $\leq 2 \cdot \frac{3L}{2} = 3L$ from $\Pi_{\mathcal{J}^c} \mathbf{u}$. Then a straightforward induction in $N \geq 2$ shows that $\text{diam} \Pi \mathbf{u} \leq (N-1) \cdot 3L < 3NL$, contrary to our hypothesis.

Now, as $\Pi \mathcal{B}(\mathbf{u}, 3L/2)$ is disconnected, there exists a nontrivial decomposition $\{1, \dots, N\} = \mathcal{J} \cup \mathcal{J}^c$ for which

$$\begin{aligned} d\left(\Pi_{\mathcal{J}} \mathcal{B}(\mathbf{u}, 3L/2), \Pi_{\mathcal{J}^c} \mathcal{B}(\mathbf{u}, 3L/2)\right) &\geq 1 \\ \Rightarrow d\left(\Pi_{\mathcal{J}} \mathcal{B}(\mathbf{u}, L), \Pi_{\mathcal{J}^c} \mathcal{B}(\mathbf{u}, L)\right) &> \frac{1}{2}L + \frac{1}{2}L = L, \end{aligned}$$

as asserted in Eqn (2.14). \square

The decomposition $(\mathcal{J}, \mathcal{J}^c)$ figuring in Lemma 2.6 may be not unique. We will assume that such a decomposition (referred to as the canonical one) is associated in some unique way with every N -particle WI ball. Accordingly,

we fix the notation $N' = \sharp \mathcal{J}$, $N'' = \sharp \mathcal{J}^c = N - N'$, and further – for $\mathbf{x} = (x_1, \dots, x_N) \in \mathcal{B}(\mathbf{u}, L)$ –

$$\begin{aligned} \mathbf{x}_{\mathcal{J}} &= (x_{i_1}, \dots, x_{i_{N'}}), \quad \mathbf{x}_{\mathcal{J}^c} = (x_{j_1}, \dots, x_{j_{N''}}) \\ \text{where } \mathcal{J} &= \{i_1, \dots, i_{N'}\}, \quad \mathcal{J}^c = \{j_1, \dots, j_{N''}\}, \\ \text{with } 1 \leq i_1 < \dots < i_{N'} \leq N, \quad 1 \leq j_1 < \dots < j_{N''} \leq N. \end{aligned} \quad (2.15)$$

This gives rise to the Cartesian product representation

$$\begin{aligned} \mathcal{B} &= \mathcal{B}' \times \mathcal{B}'' \quad \text{where } \mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L) \text{ and} \\ \mathcal{B}' &= \mathcal{B}^{(N')}(\mathbf{u}_{\mathcal{J}}, L), \quad \mathcal{B}'' = \mathcal{B}^{(N'')}(\mathbf{u}_{\mathcal{J}^c}, L), \end{aligned} \quad (2.16)$$

which we also call the canonical factorization.

Consequently, the operator $\mathbf{H}_{\mathcal{B}}^{(N)}$ in a WI ball $\mathcal{B} = \mathcal{B}(\mathbf{u}, L)$ can be represented in the following way:

$$\mathbf{H}_{\mathcal{B}}^{(N)} = \mathbf{H}_{\mathcal{B}'}^{(N')} \otimes \mathbf{I}^{(N'')} + \mathbf{I}^{(N')} \otimes \mathbf{H}_{\mathcal{B}''}^{(N'')} + \mathbf{U}_{\mathcal{B}', \mathcal{B}''}. \quad (2.17)$$

Here the summand $\mathbf{U}_{\mathcal{B}', \mathcal{B}''}$ takes into account the interaction between subsystems in balls \mathcal{B}' and \mathcal{B}'' and has a small norm for L large. Operators $\mathbf{H}_{\mathcal{B}'}^{(N')}$ and $\mathbf{H}_{\mathcal{B}''}^{(N'')}$ are called the reduced Hamiltonians (for the WI ball \mathcal{B}).

Lemma 2.7. *Let $\mathcal{B}^{(N)}(\mathbf{x}, L)$, $\mathcal{B}^{(N)}(\mathbf{y}, L)$ be a pair of SI balls with $\rho(\mathbf{x}, \mathbf{y}) > 8NL$. Then*

$$\Pi \mathcal{B}^{(N)}(\mathbf{x}, L) \cap \Pi \mathcal{B}^{(N)}(\mathbf{y}, L) = \emptyset, \quad (2.18)$$

and, consequently, the random operators $\mathbf{H}_{\mathcal{B}^{(N)}(\mathbf{x}, L)}(\omega)$ and $\mathbf{H}_{\mathcal{B}^{(N)}(\mathbf{y}, L)}(\omega)$ are independent.

Proof. By definition, for any SI balls $\mathcal{B}^{(N)}(\mathbf{x}, L)$, $\mathcal{B}^{(N)}(\mathbf{y}, L)$ we have

$$\max_{i,j} d(x_i, x_j) \leq 3NL, \quad \max_{i,j} d(y_i, y_j) \leq 3NL,$$

and it follows from the assumption $\rho(\mathbf{x}, \mathbf{y}) > ANL$ that for some $i', j' \in \{1, \dots, N\}$ $d(x_{i'}, y_{j'}) > 3NL$, thus for any $i, j \in \{1, \dots, N\}$

$$d(x_i, y_j) \geq d(x_{i'}, y_{j'}) - d(x_{i'}, x_i) - d(x_{j'}, y_j) > 8NL - 6NL = 2NL.$$

Therefore, with $N \geq 1$,

$$\text{dist}(\Pi \mathcal{B}_L(\mathbf{x}), \Pi \mathcal{B}_L(\mathbf{y})) > 2(N-1)L \geq 0,$$

so $\Pi \mathcal{B}^{(N)}(\mathbf{x}, L) \cap \Pi \mathcal{B}^{(N)}(\mathbf{y}, L) = \emptyset$. Consequently, the samples of the random potential in $\mathbf{H}_{\mathcal{B}^{(N)}(\mathbf{x}, L)}(\omega)$ and $\mathbf{H}_{\mathcal{B}^{(N)}(\mathbf{y}, L)}(\omega)$ are independent. \square

Throughout the paper we consider a sequence of integers $L_k > 1$ of one of the two forms

$$(a) \quad L_{k+1} := L_k B, \quad k = 0, 1, \dots, \quad \text{or} \quad (b) \quad L_{k+1} = \lfloor L_k^\alpha \rfloor, \quad k = 0, 1, \dots \quad (2.19)$$

with given initial positive integer values L_0, B and a scaling exponent $\alpha > 1$. Referring to (2.19), we consider

Definition 2.4. A ball $\mathcal{B}^{(N)}(\mathbf{x}, L_k)$, $k \geq 1$, is called (E, β) -completely non-resonant $((E, \beta)$ -CNR) if all concentric balls $\mathcal{B}^{(N)}(\mathbf{x}, \ell)$ with $L_{k-1} \leq \ell \leq L_k$ are (E, β) -NR.

The next definition is based upon Definitions 2.1 and 2.4. Here we use parameter m_N of the following form:

$$m_N := m^* (1 + 3L_0^{-\delta+\beta})^{N^*-N+1} \quad (2.20)$$

with some $\beta \in (0, 1)$, $\delta \in (\beta, 1)$ and $m^* > 0$. Note that for all $N = 1, \dots, N^* - 1$, we have (cf. (3.1))

$$m_N = (1 + 4L_0^{-\delta+\beta})m_{N+1} > m_{N+1} \geq m_*.$$

Definition 2.5. Assume that $\beta \in (0, 1)$, $\delta \in (0, 1]$ and ν_N are as in (3.1). Next, let $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L)$ be a WI ball with the canonical factorization (2.16) and suppose that, for a given $E \in \mathbb{R}$, \mathcal{B} is (E, β) -NR. The following properties (i), (ii) are defined in terms of the reduced Hamiltonians $\mathbf{H}_{\mathcal{B}'}^{(N')}$ and $\mathbf{H}_{\mathcal{B}''}^{(N'')}$. We say that

(i) \mathcal{B} is (E, β) -fully non-resonant $((E, \beta)$ -FNR) if

$$\begin{aligned} & \forall \lambda' \in \Sigma \left(\mathbf{H}_{\mathcal{B}'}^{(N')} \right), \text{ ball } \mathcal{B}'' \text{ is } (E - \lambda', \beta)\text{-CNR} \\ & \text{and} \\ & \forall \lambda'' \in \Sigma \left(\mathbf{H}_{\mathcal{B}''}^{(N'')} \right), \text{ ball } \mathcal{B}' \text{ is } (E - \lambda'', \beta)\text{-CNR}. \end{aligned} \quad (2.21)$$

Furthermore, we say that

(ii) \mathcal{B} is $(E, \delta, \nu_{N'}, \nu_{N''})$ -partially non-singular $((E, \delta, m_{N'}, m_{N''})$ -PNS) if in Eqns (2.21) we have properties $(E - \lambda', \delta, m_{N'})$ -NS and $(E - \lambda'', \delta, m_{N''})$ -NS, instead of $(E - \lambda', \beta)$ -CNR and $(E - \lambda'', \beta)$ -CNR, respectively:

$$\begin{aligned} & \forall \lambda' \in \Sigma \left(\mathbf{H}_{\mathcal{B}'}^{(N')} \right), \text{ ball } \mathcal{B}'' \text{ is } (E - \lambda', \delta, m_{N'})\text{-NS} \\ & \text{and} \\ & \forall \lambda'' \in \Sigma \left(\mathbf{H}_{\mathcal{B}''}^{(N'')} \right), \text{ ball } \mathcal{B}' \text{ is } (E - \lambda'', \delta, m_{N''})\text{-NS}. \end{aligned} \quad (2.22)$$

Property $(E, \delta, m_{N'}, m_{N''})$ -PNS is employed in Lemma 3.3 whereas (E, β) -FNR in Theorem 2.8 (and in several places later). Furthermore, in Theorem 2.8 we refer to the case (a) in Eqn (2.19):

$$L_{k+1} := L_0 B^k, \quad k = 0, 1, \dots, \quad \text{where } L_0, B \in \mathbb{N}^*, B \geq 2. \quad (2.23)$$

Theorem 2.8. Fix $b, \beta \in (0, 1)$ and $B \geq 2$. If L_0 is chosen large enough then for all $E \in \mathbb{R}$ and $k \geq 0$,

$$\mathbb{P} \left\{ \text{ball } \mathcal{B}^{(N)}(\mathbf{u}, L_k) \text{ is WI but not } (E, \beta)\text{-FNR} \right\} \leq 2e^{-(2B)^{-1}L_k^{\beta b}}. \quad (2.24)$$

Proof. Let ball $\mathcal{B}^{(N)}(\mathbf{u}, L_k)$ be WI and (E, β) -NR. To shorten the notation, set – referring to canonical factorization (2.16) –

$$\Sigma' = \Sigma \left(\mathbf{H}_{\mathcal{B}'}^{(N')} \right) \quad \text{and} \quad \Sigma'' = \Sigma \left(\mathbf{H}_{\mathcal{B}''}^{(N'')} \right). \quad (2.25)$$

Consider the event $\mathcal{S} = \{\mathcal{B} \text{ is not } (E, \beta)\text{-FNR}\}$. Then $\mathcal{S} \subset \mathcal{S}' \cup \mathcal{S}''$, where

$$\begin{aligned}\mathcal{S}' &= \{\exists \lambda'' \in \Sigma'' : \mathcal{B}' \text{ is not } (E - \lambda'', \beta)\text{-CNR}\}, \\ \mathcal{S}'' &= \{\exists \lambda' \in \Sigma' : \mathcal{B}'' \text{ is not } (E - \lambda', \beta)\text{-CNR}\}.\end{aligned}\quad (2.26)$$

First, assess $\mathbb{P}\{\mathcal{S}'\}$. Denoting by \mathfrak{F}'' the sigma-algebra generated by the values $V(x)$, $x \in \Pi_{\mathcal{J}^0} \mathcal{B}$, write:

$$\begin{aligned}\mathbb{P}\{\mathcal{S}'\} &\leq (\#\mathcal{B}'') \sum_{L_{k-1} \leq \ell \leq L_k} \max_{\lambda'' \in \Sigma''} \mathbb{E} \left[\mathbb{P}\{\mathcal{B}' \text{ is } (E - \lambda'', \beta)\text{-R} \mid \mathfrak{F}''\} \right] \\ &\leq L_k \cdot (C_{\mathbb{Z}}^N L_k^{Nd}) \sup_{E' \in \mathbb{R}} \mathbb{P}\{\mathcal{B}^{(N')}(\mathbf{u}_{\mathcal{J}}, \ell) \text{ is } (E', \beta)\text{-R}\} \\ &\leq C_{\mathbb{Z}}^N L_k^{Nd+1} e^{-L_{k-1}^{\beta b}} = e^{-B^{-\beta b} L_k^{\beta b} + \ln(C_{\mathbb{Z}}^N L_k^{Nd+1})} \leq e^{-(2B)^{-1} L_k^{\beta b}},\end{aligned}\quad (2.27)$$

provided that L_0 is large enough, which we assumed.

As the roles of \mathcal{B}' and \mathcal{B}'' are symmetric, the same upper bound holds for $\mathbb{P}\{\mathcal{S}''\}$. This completes the proof of Theorem 2.8. \square

Theorem 2.8 will be instrumental for the proof of Theorem 3.4 and – in a form modified to case (b) in (2.19) – in the proof of Theorem 4.3.

3 Fixed- and variable-energy estimates

The aim in this section is to prove assertion (A) of Theorem 1.1; the main technical tool is provided by so-called variable-energy estimates. (In [24] the term continuum-energy has been used.) An example is Theorem 3.8. In the MPMSA, such bounds are difficult to obtain; in a sense, they represent a bottleneck of the whole method. Nevertheless, until subsection 3.3 we work with much simpler fixed-energy estimates, preparing the grounds for the passage to the variable-energy ones. The sequence $\{L_k\}$ is taken in this section of the form (2.23).

Throughout the section we use, in various combinations, inequalities listed in (3.1). These inequalities are imposed upon key parameters of the inductive schemes involved. Namely, we employ the following parameters: (i) B and L_0 (positive integers); (ii) $\kappa \in (0, \zeta]$ (bounds the decay of the EFCs); (iii) $\beta \in (0, 1)$ (a resonance/nonresonance threshold value, emerging in (2.2)); (iv) $m^* \geq 1$ giving rise to a ‘mass’ m_N and $\delta \in (0, 1)$ (a sub-exponential decay parameter figuring in (2.3)); (v) K (a nonnegative integer appearing in (3.2) and controlling the number of singular balls of radius L_k inside a ball of radius L_{k+1}); (vi) $\nu^* \geq 1$ used in (3.5) through the scaled value ν_N controlling the decay of the so-called singularity probability. In the table (3.1) we show the relations between these parameters. (A specific form of some of these relations is chosen for technical convenience.) Recall, N takes values $1, \dots, N^*$.

The integer K appears in Definition 3.1 below and also in Definition 4.2 (cf. Sect. 4). In Sect. 4, the MSA induction is adapted to the length scale sequence satisfying $L_{k+1} = \lfloor L_k^\alpha \rfloor$, where $\alpha > 1$ depends upon the decay exponent $\zeta > 0$ of the interaction potential, and K is to be chosen large enough, depending upon ζ . In Sect. 3, it suffices to set $K = 1$ to obtain sub-exponential decay of EF correlators with some exponent $\kappa > 0$, but we keep the value of K in symbolic form. It is worth mentioning that by choosing $K > 1$ large enough, one can

make $\kappa \in (0, 1)$ arbitrarily close to 1, but this requires some additional analysis which we omit for brevity and clarity of presentation. For further details, see the work by Klein and Nguyen [24], adapting to the multi-particle setting the bootstrap MSA techniques, originally developed in [21].

$0 < \kappa < \zeta, 0 < \beta < \delta < \zeta \wedge 1$	$\beta + \frac{\ln(8B)}{\ln L_0} < \delta < 1 - \frac{\ln 12}{\ln B}$	(3.1)
$m^*, \nu^* \geq 1$	$B \geq 24N^*K; L_0$ large enough depending on $\beta, \delta, B, K, m^*, \nu^*$	
$m_N = m^* (1 + 4L_0^{-\delta+\beta})^{N^*-N+1}$	$\nu_N = \nu^* (2B^\kappa)^{N^*-N+1}$	

For definiteness, we assume (3.1) to be satisfied throughout the whole section 3, regardless of whether a particular parameter is involved in a given assertion or not. This will not be reminded every time again, although basic ranges for values of $\beta, \delta, \kappa, m^*, \nu^*$ will be outlined. (A number of technical statements remain valid under broader restrictions than those from (3.1).)

3.1 Scaling the GFs. Property $\mathbf{S}(N, k)$

Definition 3.1. Suppose that the following values are given: $E \in \mathbb{R}$, $\beta, \delta \in (0, 1)$, $m^* \geq 1$ and integers $k, K \geq 0$. An N -particle ball $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})$ is called (E, m_N, K) -good $((E, m_N, K)$ -G) if \mathcal{B} is (E, β) -CNR (cf. Definition 2.4) and

$$\begin{aligned} \mathcal{B} \text{ contains no collection of } \geq K+1 \text{ balls of radius } L_k \\ \text{which are pairwise } 8NL_k\text{-distant and } (E, \delta, m_N)\text{-S.} \end{aligned} \quad (3.2)$$

In this definition we omitted parameters $\delta, \beta \in (0, 1)$ from the notation (E, m_N, K) -G).

Lemma 3.1. Given β, δ, m^*, K and B , suppose that L_0 is large enough. If a ball $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})$ is (E, β) -NR and (E, m_N, K) -G, then \mathcal{B} is (E, δ, m_N) -NS. The assertion remains valid under a weaker condition than (E, β) -NR:

$$\text{dist} \left(\Sigma \left(\mathbf{H}_{\mathcal{B}}^{(N)} \right), E \right) \geq e^{-L^\beta}. \quad (3.3)$$

Proof. In this proof we use Lemmas B.1 and B.2 from Appendix B. Fix $K' \leq K$ and a maximal collection of K' pairwise $8NL_k$ -distant, (E, δ, m_N) -NS balls of radius L_k lying in \mathcal{B} . Let \mathcal{N} denote the L_k -neighborhood of the union of these balls. Then any ball $\mathcal{B}^{(N)}(\mathbf{v}, L_k)$ with $\mathbf{v} \in \mathcal{B} \setminus \mathcal{N}$ is (E, β, m_N) -NS. Bearing in mind Lemma B.2, denote by Ξ be the union of all spherical layers $\mathcal{L}_r(\mathbf{u})$ such that $\mathcal{L}_r(\mathbf{u}) \cap \mathcal{N} \neq \emptyset$. It follows from table (3.1) (the relations between δ and β) that

$$m_N - 2L_k^{-\delta} L_{k+1}^\beta = m_N \left(1 - 2m_N^{-1} L_k^{-\delta+\beta} B^\beta \right) \geq \frac{3}{4} m_N > 0. \quad (3.4)$$

Thus, by Lemma B.2, the function

$$f : \mathbf{x} \in \mathcal{B} \mapsto \left| G_{\mathcal{B}}^{(N)}(\mathbf{u}, \mathbf{x}; E) \right|$$

is (ℓ, q, Ξ) -dominated in \mathcal{B} , with $q \leq e^{-\frac{3}{4}m_N L_k^\delta}$. Cf. Eqn (B.6).

Owing to Lemma B.1, we can write, with the convention $-\ln 0 = +\infty$, that $-\ln f(\mathbf{x})$ is bounded below by

$$-\ln \left\{ e^{L_{k+1}^\beta} \exp \left[-\frac{3m_N}{4} L_k^\delta \cdot \frac{L_{k+1} - (8NK + 2)L_k - 2L_k}{L_k + 1} \right] \right\}$$

thus by virtue of conditions in Eqn (3.1) (in particular, with $L_0 \geq 3$, $B \geq 24N^*K \geq 24NK$ and $\frac{1}{4}B^{1-\delta} \geq 3$), one obtains by a simple calculation

$$-\ln f(\mathbf{x}) \geq 2m_N L_{k+1}^\delta \geq m_N L_{k+1}^\delta + \ln(C_{\mathcal{Z}}^{2N} L_{k+1}^d).$$

□

Lemma 3.1 has a multiple use: it is needed in the proof of Theorems 3.4 and 3.5.

Given $L_0, B, \delta, \kappa, m^*, \nu^*$, consider the following property $S(N, k)$ depending upon N and k (S stands for singularity):

$S(N, k)$: $\forall E \in \mathbb{R}, 1 \leq n \leq N$ and configuration $\mathbf{u} \in \mathcal{Z}$

$$\mathbb{P} \left\{ \text{ball } \mathcal{B}^{(n)}(\mathbf{u}, L_k) \text{ is } (E, \delta, m_n)\text{-}S \right\} \leq e^{-\nu_n L_k^\kappa}. \quad (3.5)$$

The MPMSA inductive scheme consists in checking $S(N, k) \forall N$ and k . The initial step of induction in k is established in Theorem 3.2.

Theorem 3.2. *Suppose a positive integer M and an $M \times M$ Hermitian matrix \mathbf{A} are given, as well as random variables (not assumed to be independent) W_1, \dots, W_M , with continuous distribution functions F_{W_i} , $1 \leq i \leq M$. Let $\mathbf{W}(\omega)$ be the diagonal random matrix $\text{diag}(W_1(\omega), \dots, W_M(\omega))$. For any $s > 0$ and $\epsilon \in (0, 1)$, there exists $g^* < \infty$ such that if $|g| \geq g^*$ then*

$$\sup_{E \in \mathbb{R}} \mathbb{P} \left\{ \|(\mathbf{A} + g\mathbf{W} - E\mathbf{I})^{-1}\| > s \right\} \leq \epsilon < 1.$$

Consequently, $\forall \delta \in (0, 1)$, $\kappa \in (0, \zeta)$ and $m^, \nu^* \geq 1$, $\exists g_0 \in (0, \infty)$ such that $\forall 1 \leq N \leq N^*$ and positive integer L_0 , property $S(N, 0)$ holds true.*

The proof is omitted; it is based on a well-known argument employed in a number of papers on the MSA (cf., e.g., [18, Proposition A.1.2]) and is not contingent upon the single- or multi-particle structure of the random diagonal entries of the matrix \mathbf{A} .

3.2 The GFs in WI balls. The MPMSA induction

Lemma 3.3. *Fix $\beta, \delta \in (0, 1)$, $m^* \geq 1$ and an energy value $E \in \mathbb{R}$. Consider a WI N -particle ball $\mathcal{B}^{(N)}(\mathbf{u}, L_k)$ with a canonical factorization $\mathcal{B}^{(N')}(\mathbf{u}_{\mathcal{J}}, L_k) \times \mathcal{B}_L^{(N'')}(\mathbf{u}_{\mathcal{J}^c}, L_k)$. Suppose that $\mathcal{B}^{(N)}(\mathbf{u}, L_k)$ is (E, β) -NR and $(E, \delta, m_{N'}, m_{N''})$ -PNS. Cf. Definition 2.5, (ii). If L_0 is large enough then $\mathcal{B}^{(N)}(\mathbf{u}, L_k)$ is (E, δ, m_N) -NS.*

Proof. See Appendix A.1. \square

Lemma 3.3 is used in the proof of Theorem 3.4.

Theorem 3.4. *Assume property $S(N-1, k)$ for some given $L_0, B > 1$, $\delta \in (0, 1)$, $\kappa \in (0, \zeta)$ and $m^*, \nu^* \geq 1$ (see Eqn (3.5)). Then if L_0 is large enough then for any $E \in \mathbb{R}$ and WI ball $\mathcal{B}^{(N)}(\mathbf{u}, L_k)$,*

$$\mathbb{P} \left\{ \mathcal{B}^{(N)}(\mathbf{u}, L_k) \text{ is } (E, \delta, m_N)\text{-S} \right\} \leq 2e^{-\frac{3}{2}\nu_N L_{k+1}^\kappa}. \quad (3.6)$$

Consequently, for L_0 large enough, $\forall \mathbf{x} \in \mathcal{Z}$,

$$\begin{aligned} \mathbb{P} \left\{ \mathcal{B}^{(N)}(\mathbf{x}, L_{k+1}) \text{ contains a WI } (E, \delta, m_N)\text{-S ball } \mathcal{B}^{(N)}(\mathbf{u}, L_k) \right\} \\ \leq C_{\mathcal{Z}}^N L_{k+1}^{Nd} \cdot 2e^{-\frac{3}{2}\nu_N L_{k+1}^\kappa} \leq \frac{1}{4} e^{-\nu_N L_{k+1}^\kappa}. \end{aligned} \quad (3.7)$$

Proof. Denote by \mathcal{S} the event in the LHS of (3.6). Set $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L_k)$ and write the canonical factorization $\mathcal{B} = \mathcal{B}' \times \mathcal{B}''$ with reduced operators $\mathbf{H}' = \mathbf{H}_{\mathcal{B}'}^{(N')}$ and $\mathbf{H}'' = \mathbf{H}_{\mathcal{B}''}^{(N'')}$ (cf. (2.16), (2.17)). By Lemma 3.1,

$$\begin{aligned} \mathbb{P} \{ \mathcal{S} \} &< \mathbb{P} \{ \mathcal{B} \text{ is not } E\text{-FNR} \} \\ &+ \mathbb{P} \{ \mathcal{B} \text{ is } E\text{-FNR and } (E, \delta, m_N)\text{-S} \}. \end{aligned} \quad (3.8)$$

The first term in the RHS is assessed in Theorem 2.8, so we focus on the second summand. Apply Lemma 3.3 and introduce events \mathcal{S}' and \mathcal{S}'' by following the framework of Eqn (2.26) and (2.27). Then, with $m'' = m_{N''}$,

$$\mathbb{P} \{ \mathcal{S}' \} = \mathbb{E} \left[\mathbb{P} \left\{ \exists \lambda'' \in \Sigma(\mathbf{H}'') : \mathcal{B}' \text{ is } (E - \lambda'', \delta, m'')\text{-S} \mid \mathfrak{F}'' \right\} \right].$$

By definition of the canonical decomposition, $\Pi \mathcal{B}' \cap \Pi \mathcal{B}'' = \emptyset$, and since the random field V is IID, for any $E'' \in \mathbb{R}$, including $E - \lambda''$, the conditional probability does not depend on the condition:

$$\mathbb{P} \{ \mathcal{B}' \text{ is } (E'', \delta, m'')\text{-S} \mid \mathfrak{F}'' \} \stackrel{\text{a.s.}}{=} \mathbb{P} \{ \mathcal{B}' \text{ is } (E'', \delta, m'')\text{-S} \}. \quad (3.9)$$

On the other hand, by virtue of the (assumed) property $S(N-1, k)$, for $N' \leq N-1$,

$$\mathbb{P} \{ \mathcal{B}' \text{ is } (E'', \delta, m'')\text{-S} \} \leq e^{-\nu_{N-1} L_k^\kappa} = e^{-2\nu_N L_{k+1}^\kappa}. \quad (3.10)$$

Thus, in analogy with (2.27), we obtain that

$$\begin{aligned} \mathbb{P} \{ \mathcal{S}' \} &\leq \# \mathcal{B}'' \sup_{E'' \in \mathbb{R}} \mathbb{P} \{ \mathcal{B}' \text{ is } (E'', \delta, m'')\text{-S} \} \\ &\leq C_{\mathcal{Z}}^N L_k^{Nd} \exp \{ -2\nu_N L_{k+1}^\kappa \} \leq \exp \left\{ -\frac{3}{2}\nu_N L_{k+1}^\kappa \right\}; \end{aligned} \quad (3.11)$$

here the last inequality holds for L_0 large enough. Similarly, with $m' = m_{N'}$,

$$\begin{aligned} \mathbb{P} \{ \mathcal{S}'' \} &= \mathbb{E} \left[\mathbb{P} \left\{ \exists \lambda' \in \Sigma(\mathbf{H}') : \mathcal{B}'' \text{ is } (E - \lambda', \delta, m')\text{-S} \mid \mathfrak{F}' \right\} \right] \\ &\leq \exp \left\{ -\frac{3}{2}\nu_N L_{k+1}^\kappa \right\}. \end{aligned} \quad (3.12)$$

Collecting (2.4), (3.8), (3.11) and (3.12), the assertion (3.6) follows.

To prove (3.7), notice that the number of WI balls of radius L_k inside $\mathcal{B}^{(N)}(\mathbf{x}, L_{k+1})$ is bounded by the cardinality $\sharp \mathcal{B}^{(N)}(\mathbf{x}, L_{k+1})$, and the probability for a WI ball to be (E, δ, m_N) -S satisfies (3.6), so the last inequality in (3.7) follows, again for L_0 large enough. \square

Now, given $k = 0, 1, \dots$, consider the following probabilities:

$$\begin{aligned} P_k &= \sup_{\mathbf{u} \in \mathcal{Z}} \mathbb{P} \left\{ \text{ball } \mathcal{B}^{(N)}(\mathbf{u}, L_k) \text{ is } (E, \delta, m)\text{-S} \right\}, \\ Q_{k+1} &= 4 \sup_{\mathbf{u} \in \mathcal{Z}} \mathbb{P} \left\{ \mathcal{B}^{(N)}(\mathbf{u}, L_{k+1}) \text{ is } (E, \beta)\text{-R} \right\}, \\ S_{k+1} &= \sup_{\mathbf{x} \in \mathcal{Z}^N} \mathbb{P} \left\{ \mathcal{B}^{(N)}(\mathbf{x}, L_{k+1}) \text{ contains a WI } (E, \delta, m)\text{-S ball } \mathcal{B}^{(N)}(\mathbf{u}, L_k) \right\}. \end{aligned}$$

Note that

$$\text{for } N = 1, S_{k+1} = 0 \text{ (there are no WI balls).} \quad (3.13)$$

Theorem 3.5. *Suppose that, for some given $B \geq 2$, $\delta \in (0, 1)$, $\kappa \in (0, \zeta)$ and $m^*, \nu^* \geq 1$, property $S(N, 0)$ holds true with L_0 large enough. Then $S(N, k)$ holds true $\forall k \geq 0$ with the same $L_0, B, \delta, \kappa, m^*$ and ν^* .*

Proof. It suffices to derive $S(N, k+1)$ from $S(N, k)$, so assume the latter. By virtue of Lemma 3.1, if a ball $\mathcal{B}^{(N)}(\mathbf{x}, L_{k+1})$ is (E, δ) -S, then it is either (E, β) -R (with probability $\leq \frac{1}{4}Q_{k+1}$) or (E, ν, K) -bad.

By Eqn (3.7), the probability of having at least one WI (E, m_N) -S ball $\mathcal{B}^{(N)}(\mathbf{u}, L_k) \subset \mathcal{B}^{(N)}(\mathbf{x}, L_{k+1})$ obeys: $S_{k+1} \leq \frac{1}{4}e^{-\nu_N L_{k+1}^\kappa}$.

Note that this is the only point where the inductive hypothesis $S(N-1, k)$ is actually required, for $N \geq 2$, while $S_{k+1} = 0$ for $N = 1$, because of (3.13).

Therefore, it remains to assess the probability of having a collection of at least K balls of radius L_k inside $\mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})$ which are SI, (E, δ) -S and pairwise $8NL$ -distant. The number of such collections is $\leq C_{\mathcal{Z}}^{KN} L_{k+1}^{KNd}$, thus, owing to Lemma 2.7, we have¹

$$P_{k+1} \leq \frac{1}{2} C_{\mathcal{Z}}^{2N} L_{k+1}^{KNd} P_k^{K+1} + S_{k+1} + \frac{1}{4} Q_{k+1}.$$

By Theorem 2.1, $Q_{k+1} \leq C_W L_k^{(N+1)d} e^{-L_{k+1}^\beta}$, $\beta > \kappa$, so with L_0 large enough, $Q_k \leq \frac{1}{4}e^{-\nu_N L_k^\kappa}$ for any $k \geq 0$.

Finally,

$$P_{k+1} \leq \frac{1}{2} C_{\mathcal{Z}}^{KN} L_{k+1}^{KNd} P_k^{K+1} + S_{k+1} + \frac{1}{4} Q_{k+1} \quad (3.14)$$

with $S_{k+1} + \frac{1}{4}Q_{k+1} \leq \frac{1}{4}e^{-\nu_N L_k^\kappa} + \frac{1}{4}e^{-\nu_N L_k^\kappa} \leq \frac{1}{2}e^{-\nu_N L_k^\kappa}$. The assertion of the theorem will follow if we show that, for L_0 large enough, $P_{k+1} \leq e^{-\nu_N L_k^\kappa}$, i.e., $C_{\mathcal{Z}}^{KN} L_{k+1}^{KNd} P_k^{K+1} \leq e^{-\nu_N L_k^\kappa}$. The last fact can be verified by using inequalities (3.1). This completes the proof of Theorem 3.5. \square

Theorem 3.5 allows us to complete the MPMSA inductive scheme.

In fact, owing to Theorem 3.2, \forall given $L_0, B, \delta, \kappa, m^*$ and ν^* , property $S(N, 0)$ holds true for sufficiently large $|g|$ and all $N = 1, \dots, N^*$. The scale

¹A statement like Lemma 2.7 is not required for $N = 1$, if the random potential is IID, since the disjoint 1-particle balls give rise to independent Hamiltonians.

induction step $k \rightsquigarrow k+1$ is provided by Lemmas 3.1 and 3.3 and Theorems 3.4 and 3.5. By induction in k , this proves $S(N, k)$ for $1 < N \leq N^*$ and all $k \geq 0$, provided that $S(N-1, k)$ is proved for all $k \geq 0$. The base of induction in N , is obtained in a similar (in fact, simpler) manner.

3.3 From fixed to variable energy estimates

The core of the technical argument in this section is the two-volume EVC bound (Theorem 2.2).

Given a positive integer L and $\mathbf{u} \in \mathcal{Z}$, define the quantity $\mathbf{F}_{\mathbf{u}}(E) = \mathbf{F}_{\mathbf{u}, L}^{(N)}(E)$:

$$\mathbf{F}_{\mathbf{u}}(E) = C_{\mathcal{Z}}^{2N} L^{Nd} \max_{\mathbf{z} \in \partial^- \mathcal{B}(\mathbf{x}, L)} |G_{\mathcal{B}(\mathbf{u}, L)}^{(N)}(\mathbf{u}, \mathbf{z}; E)|. \quad (3.15)$$

For brevity, we denote by $\Sigma_{\mathbf{u}}$ the spectrum of the operator $\mathbf{H}_{\mathcal{B}(\mathbf{u}, L)}^{(N)}$. Owing to assumption (V), the norms of the operators $\mathbf{H}_{\mathcal{B}(\mathbf{u}, L)}^{(N)}$ are uniformly bounded, and so are their spectra. Therefore, the spectral analysis of these operators can be restricted, without loss of generality, to some finite interval $I \subset \mathbb{R}$ independent of $k \geq 0$ and $N = 1, \dots, N^*$.

Theorem 3.6 encapsulates a probabilistic estimate essentially going back to the work [19]. Its general strategy (converting fixed-energy probabilistic bounds into those on the measure of "resonant" energies, with the help of the Chebyshev inequality) was employed earlier in [26], and – in a different context – in [5]. Here we follow closely the book [17].

Theorem 3.6 (Cf. [17, Theorems 2.5.1 and 4.3.11]). *Let be given an integer $L \geq 1$, balls $\mathcal{B}^{(N)}(L, \mathbf{x})$, $\mathcal{B}^{(N)}(L, \mathbf{y})$, an interval $I \subset \mathbb{R}$ of length $|I| < \infty$ and numbers $a_L, b_L, c_L, q_L > 0$ satisfying*

$$b_L \leq \min\{K^{-1}a_L c_L^2, c_L\}. \quad (3.16)$$

with $K = \max \left[\#\mathcal{B}^{(N)}(L, \mathbf{x}), \#\mathcal{B}^{(N)}(L, \mathbf{y}) \right]$. Suppose in addition that

$$\mathbb{P}\{\mathbf{F}_{\mathbf{x}}(E) \geq a_L\} \leq q_L, \quad \mathbb{P}\{\mathbf{F}_{\mathbf{y}}(E) \geq a_L\} \leq q_L. \quad (3.17)$$

Assume also that for some $A, C, > 0$ and $\theta \in (0, 1]$, $\forall \epsilon > 0$

$$\mathbb{P}\left\{\text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq \epsilon\right\} \leq CL^A \epsilon^\theta. \quad (3.18)$$

Then one has, with $A' = A + 2Nd$ and some $C' < \infty$,

$$\mathbb{P}\left\{\exists E \in I : \min[\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)] \geq a_L\right\} \leq \frac{2|I|q_L}{b} + C' L^{A'} c_L^\theta. \quad (3.19)$$

To prove Theorem 3.6, we need a general result, Theorem 3.7. A similar assertion had been proven for balls in an integer lattice \mathbb{Z}^d (cf. Theorem 2.5.1 in [17]). A direct inspection shows that the assertion remains true for $\mathcal{Z} = \mathcal{Z}^N$ (with metric ρ), and any finite subset $\mathcal{V} \subset \mathcal{Z}$. Here we state it for a ball $\mathcal{B}(\mathbf{u}, L)$, only to set up a framework for the proof of Theorem 3.6.

Theorem 3.7 (Cf. Theorem 2.5.1 in [17]). *Given a positive integer L and a configuration $\mathbf{u} \in \mathcal{Z}$, take the ball $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L)$. Set $K = \sharp \mathcal{B}$ and consider a random operator $\mathbf{H}_{\mathcal{B}} = \mathbf{H}_{\mathcal{B}}^{(N)}(\omega)$ of the form*

$$(\mathbf{H}_{\mathcal{B}} f)(\mathbf{x}) = (-\Delta_{\mathcal{B}} f)(\mathbf{x}) + W(\mathbf{x}; \omega) f(\mathbf{x}), \quad \mathbf{x} \in \mathcal{B}, \quad (3.20)$$

where $(\mathbf{x}, \omega) \mapsto W(\mathbf{x}; \omega) \in \mathbb{R}$ is a given random function. (No condition is imposed upon the distribution of $W(\mathbf{x}; \omega)$.) Let $E_j = E_j(\omega)$, $1 \leq j \leq K$, be the (random) eigenvalues of $\mathbf{H}_{\mathcal{B}}$ listed in some measurable way. Take a bounded interval $I \subset \mathbb{R}$ and let the numbers $a_L, b_L, c_L, q_L > 0$ satisfy

$$b_L \leq \min\{K^{-1}a_L c_L^2, c_L\} \quad (3.21)$$

and $\forall E \in I$

$$\mathbb{P}\{\mathbf{F}_{\mathbf{u}}(E) \geq a_L\} \leq q_L \quad (3.22)$$

where $\mathbf{F}_{\mathbf{u}}(E)$ is as in Eqn (3.15). Then there is an event \mathcal{C} of probability $\mathbb{P}\{\mathcal{C}\} \leq |I|b_L^{-1}q_L$ such that for all $\omega \notin \mathcal{C}$

$$T_{\mathbf{u}}(2a_L) := \{E \in I : \mathbf{F}_{\mathbf{u}}(E) \geq 2a_L\} \subset \cup_{j=1}^K I_j, \quad (3.23)$$

where $I_j := (E_j - 2c_L, E_j + 2c_L)$.

Proof of Theorem 3.6. Let $\mathcal{C}_{\mathbf{x}}$ and $\mathcal{C}_{\mathbf{y}}$ be the events introduced in Theorem 3.7 for the balls $\mathcal{B}^{(N)}(L, \mathbf{x})$ and $\mathcal{B}^{(N)}(L, \mathbf{y})$, and let $\mathcal{C} = \mathcal{C}_{\mathbf{x}} \cup \mathcal{C}_{\mathbf{y}}$. As in Theorem 3.7, denote by $T_{\mathbf{x}}(2a_L)$, $T_{\mathbf{y}}(2a_L)$ the energy sets related to $\mathcal{B}^{(N)}(L, \mathbf{x})$, $\mathcal{B}^{(N)}(L, \mathbf{y})$, and introduce the event $\mathcal{E} = \{\omega : T_{\mathbf{x}}(2a_L) \cap T_{\mathbf{y}}(2a_L) \neq \emptyset\}$. Then we have

$$\mathbb{P}\{\mathcal{E}\} \leq \mathbb{P}\{\mathcal{C}\} + \mathbb{P}\{\mathcal{E} \cap \mathcal{C}^c\} \leq 2b_L^{-1}q_L|I| + \mathbb{P}\{\mathcal{E} \cap \mathcal{C}^c\}. \quad (3.24)$$

For any $\omega \in \mathcal{C}^c$, each of the sets $\mathcal{E}_{\mathbf{x}}(2a)$, $\mathcal{E}_{\mathbf{y}}(2a)$ is covered by intervals of length $4c_L$ centered at the respective EVs of $\mathbf{H}_{\mathcal{B}^{(N)}(L, \mathbf{x})}^{(N)}$ and $\mathbf{H}_{\mathcal{B}^{(N)}(L, \mathbf{y})}^{(N)}$.

Recall that we have assumed the two-volume Wegner-type estimate (3.18) (with an exponent $\theta > 0$) for the pair $\mathcal{B}^{(N)}(L, \mathbf{x})$, $\mathcal{B}^{(N)}(L, \mathbf{y})$. (**N.B.** By Theorem 2.2, we actually have $\theta = 2/3$ for the N -particle Hamiltonians satisfying Assumption (V), but here we keep a more general assumption (3.18) and do not specify the value of θ .)

Thus, with $\Sigma_{\mathbf{x}} = \Sigma(\mathbf{H}_{\mathcal{B}^{(N)}(\mathbf{x}, L)}^{(N)})$ and $\Sigma_{\mathbf{y}} = \Sigma(\mathbf{H}_{\mathcal{B}^{(N)}(\mathbf{y}, L)}^{(N)})$,

$$\begin{aligned} \mathbb{P}\{\mathcal{E} \cap \mathcal{C}^c\} &\leq \mathbb{P}\{\text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq 4c_L\} \\ &\leq (\sharp \mathcal{B}(\mathbf{x}, L)) \cdot (\sharp \mathcal{B}(\mathbf{y}, L)) \cdot CL^A (2 \cdot 4c_L)^\theta \leq \overline{C} L^{2Nd+A} c_L^\theta, \end{aligned} \quad (3.25)$$

for some constant C' and with $A' = A + 2Nd$, as required. Collecting (3.24) and (3.25), the assertion of Theorem 3.6 follows. \square

We use Theorem 3.6 in the proof of Theorem 3.8.

Theorem 3.8. *Assuming L_0 large enough, $\forall k$ and a pair of $3NL_k$ -distant balls $\mathcal{B}^{(N)}(\mathbf{x}, L_k)$, $\mathcal{B}^{(N)}(\mathbf{y}, L_k)$, the following bound holds true:*

$$\mathbb{P}\left\{\exists E \in \mathbb{R} : \min\{\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)\} > e^{-m_N L_k^\delta}\right\} \leq e^{-\frac{1}{9}\nu_N L_k^\kappa}.$$

Proof. As was noted, the spectrum $\Sigma_{\mathbf{x}}$ is contained in a fixed bounded interval $I \subset \mathbb{R}$. Given $k \geq 0$, we have, owing to property $\mathbf{S}(N, k)$, that, with $a = e^{-m_N L_k^\delta}$,

$$\mathbb{P}\{\mathbf{F}_{\mathbf{x}}(E) > a\}, \mathbb{P}\{\mathbf{F}_{\mathbf{y}}(E) > a\} \leq e^{-\nu_N L_k^\kappa}.$$

The LHS of inequality (3.18) can be assessed by virtue of Theorem 2.2:

$$\mathbb{P}\{\text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq \epsilon\} \leq CL^{(2N+1)d} \epsilon$$

where $C = C_V^{(2)}$ is a constant. Next, we are going to use Theorem 3.6, with $L = L_k$ and

$$\begin{aligned} a_L &= e^{-\nu_N L^\kappa/3}, \quad b_L = e^{-2\nu_N L^\kappa/3}, \quad c_L = e^{-\nu_N L^\kappa/8}, \quad q_L = e^{-\nu_N L^\kappa}, \\ A &= 2N + 1, \quad C = C_V^{(2)}, \quad \theta = 1. \end{aligned} \quad (3.26)$$

We obtain that

$$\begin{aligned} &\mathbb{P}\left\{\sup_{E \in I} \min[\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)] > e^{-m_N L_k^\delta}\right\} \\ &\leq 2|I| \exp\left(-\frac{\nu_N}{3} L_k^\kappa\right) + \overline{C} L^{(4N+1)d} \exp\left(-\frac{\nu_N}{8} L_k^\kappa\right). \end{aligned}$$

For L_0 large enough, the RHS can be made $\leq e^{-\nu_N L_k^\kappa/9}$. The assertion of Theorem 3.8 now follows. \square

Theorem 3.8 is important for the proof of Theorem 3.10.

3.4 Strong dynamical localization

In this section we complete the proof of assertion (A) of Theorem 1.1. The staple here is Theorem 3.9 presenting a general result, under the key assumption (3.29).

Given a finite subset $\mathcal{V} \subset \mathcal{Z}$, we deal with a random Hamiltonian $\mathbf{H}_{\mathcal{V}} = \mathbf{H}_{\mathcal{V}}^{(N)}(\omega)$ on \mathcal{V} :

$$(\mathbf{H}_{\mathcal{V}} f)(\mathbf{x}) = (-\Delta_{\mathcal{V}} f)(\mathbf{x}) + W(\mathbf{x}; \omega) f(\mathbf{x}), \quad \mathbf{x} \in \mathcal{V}. \quad (3.27)$$

Here $(\mathbf{x}, \omega) \mapsto W(\mathbf{x}, \omega)$ is a bounded real-valued random field on \mathcal{V} . (As in Theorem 3.7, no assumption is made about the distribution of $W(\mathbf{x}, \omega)$.) At the same time, we consider Hamiltonians $\mathbf{H}_{\mathcal{B}(\mathbf{u}, L)}^{(N)}$ in balls $\mathcal{B}^{(N)}(\mathbf{u}, L) \subseteq \mathcal{V}$. (Treating them as restrictions of operator $\mathbf{H}_{\mathcal{V}}$ to $\ell_2(\mathcal{B}^{(N)}(\mathbf{u}, L)) \subseteq \ell_2(\mathcal{V})$.) Like before, $G_{\mathcal{V}}^{(N)}(\mathbf{u}, \mathbf{v}; E)$ stands for the (random) GF $\left\langle \mathbf{1}_{\mathbf{u}}, \left(\mathbf{H}_{\mathcal{V}}^{(N)} - E\mathbf{I}\right)^{-1} \mathbf{1}_{\mathbf{v}} \right\rangle$. Finally, as in Eqn (3.15),

$$\mathbf{F}_{\mathbf{u}}(E) = C_{\mathcal{Z}}^{2N} L^d \max_{\mathbf{z} \in \partial^- \mathcal{B}(\mathbf{u}, L)} |G_{\mathcal{B}(\mathbf{u}, L)}(\mathbf{u}, \mathbf{z}; E)|. \quad (3.28)$$

Like before, denote by $\mathcal{B}_1(\mathbb{R})$ the set of all Borel functions $\phi : \mathbb{R} \rightarrow \mathbb{C}$ with $\|\phi\|_{\infty} \leq 1$.

Theorem 3.9. (Cf. Lemma 9 in [8]) Assume that \forall positive integer L , the following bound holds true for any pair of disjoint balls $\mathcal{B}(\mathbf{x}, L), \mathcal{B}(\mathbf{y}, L) \subset \mathcal{S}$ and some positive functions u, h :

$$\mathbb{P} \{ \exists E \in \mathbb{R} : \min [\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)] > u(L) \} \leq h(L). \quad (3.29)$$

Then for any finite connected $\mathcal{V} \subset \mathcal{S}$ such that $\mathcal{V} \supset \mathcal{B}(\mathbf{x}, L) \cup \mathcal{B}(\mathbf{y}, L)$,

$$\mathbb{E} \left[\sup_{\phi \in \mathcal{B}_1(\mathbb{R})} |\langle \mathbf{1}_{\mathbf{x}} | \phi(\mathbf{H}_{\mathcal{V}}) | \mathbf{1}_{\mathbf{y}} \rangle| \right] \leq 4u(L) + h(L). \quad (3.30)$$

Proof. The proof repeats verbatim that of Lemma 9 in [8], except for the quantity $u(L)$ replacing an explicit expression e^{-mL} . \square

Recall that $\kappa < \delta$ (cf. (3.1)). For a finite \mathcal{V} , Assertion (A) follows from

Theorem 3.10. Given $M > 0$, $\exists g_* = g_*(M) \in (0, \infty)$ and $C_* = C_*(M) \in (0, \infty)$ such that, for $|g| \geq g_*(M)$, and $1 \leq N \leq N^*$, $\forall \mathbf{x}, \mathbf{y} \in \mathcal{Z}$ and a finite $\mathcal{V} \subset \mathcal{Z}$ with $\mathcal{V} \ni \mathbf{x}, \mathbf{y}$,

$$\Upsilon_{\mathbf{x}, \mathbf{y}} := \mathbb{E} \left[\sup_{\phi \in \mathcal{B}_1(\mathbb{R})} |\langle \mathbf{1}_{\mathbf{x}} | \phi(\mathbf{H}_{\mathcal{V}}) | \mathbf{1}_{\mathbf{y}} \rangle| \right] \leq C_* e^{-M(\rho_S(\mathbf{x}, \mathbf{y}))^\kappa}. \quad (3.31)$$

Proof. Without loss of generality, it suffices to prove the assertion for the pairs of points with $\rho_S(\mathbf{x}, \mathbf{y}) > 3NL_0$. Indeed, the EFC correlator is always bounded by 1, so for pairs \mathbf{x}, \mathbf{y} with $\rho_S(\mathbf{x}, \mathbf{y}) \leq 3NL_0$ the bound in (3.31) can be attained by taking a sufficiently large constant C_* .

Thus, fix points $\mathbf{x}, \mathbf{y} \in \mathcal{Z}$ with $\rho := \rho_S(\mathbf{x}, \mathbf{y}) > 3NL_0$. There exists k such that $\rho \in (3NL_k, 3NL_{k+1}]$. Arguing as above, it suffices to consider a finite $\mathcal{V} \subset \mathcal{Z}$ such that $\mathcal{B}^{(N)}(\mathbf{x}, L_k) \cup \mathcal{B}^{(N)}(\mathbf{y}, L_k) \subset \mathcal{V}$.

By Theorem 3.8 combined with Theorem 3.9, we have

$$\Upsilon_{\mathbf{x}, \mathbf{y}} \leq 4e^{-m_N L_k^\delta} + e^{-\nu_N L_k^\kappa/9}. \quad (3.32)$$

Since $\rho \leq 3NL_{k+1} = 3NBL_k$, we have that $L_k \geq \rho/(3NB)$. With $|g|$ large enough, the initial scale estimate $S(N, 0)$ in (3.5) is fulfilled with $m_N \geq 2BNM$, $\nu_N \geq 10NBM$ (cf. Theorem 3.2). Thus, with L_0 large enough,

$$\Upsilon_{\mathbf{x}, \mathbf{y}} \leq 5e^{-10NBM\rho^\kappa/(9NB)} \leq e^{-M\rho^\kappa}.$$

This completes the proof of Theorem 3.10. \square

The case of an infinite \mathcal{V} requires an additional limiting procedure (making use of the Fatou lemma applied to the EF correlators), developed in [1, Sect. 2]. As the argument can be repeated here without any significant change, we omit it from the paper.

4 Exponential decay of eigenfunctions

The aim of this section is to prove assertion (B) of Theorem 1.1, about an exponential decay of the EFs. This is achieved along a scheme developed in

[14–17] and modified to include the case of a graph $\mathcal{Z} \in \mathfrak{G}(d, C)$ and an infinite-range interaction potential U . (We also use the same terminology.) In short, the exponential decay follows when the inductive MPMSA scheme is successfully completed keeping the mass parameter $m > 0$; cf. (1.12). In this section a particular form of $m = m_N$ is used: see (4.2).

Compared to the scheme used in section 3, the main distinction is that here we adopt a super-exponential scaling scheme where

$$L_{k+1} = \lfloor L_k^\alpha \rfloor; \quad (4.1)$$

with the exponent α satisfying the conditions (4.2); it depends upon the value of ζ in in condition (U) (cf. Eqn (1.10)). (The smaller $\zeta > 0$, the larger is α .)

The property $S(N, k)$ will be replaced in this section by its counterpart, $S_{\text{exp}}(N, k)$, presented in Eqn (4.5), adapted to the exponential decay bounds. The verification of $S_{\text{exp}}(1, k)$ (a one-particle case) is done in a standard way.

The relations between various parameters involved in the MSA inductive scheme under Eqn (4.1) are summarised as follows.

$\tau > \max \left[\zeta^{-1}, 1 + \frac{\ln(3N)}{\ln L_0} \right]$	$0 < \beta < \zeta \wedge 1$	(4.2)
$\max \left(\tau, \frac{3}{2} \right) < \alpha < \frac{7}{8\beta}$	L_0 large enough, depending upon $\alpha, \beta, \tau, K, m^*, \nu^*$	
$m_N = m^* (1 + 4L_0^{-\delta+\beta})^{N^*-N+1}$	$m^* \geq 1$	
$P(N) = P^* (2\alpha)^{N^*-N+1}$	$P^* > 4N^* d\alpha$	

Like before, we assume the conditions (4.2) throughout the whole section.

4.1 The analytic step: scaling the GFs

We continue our analysis of GFs of $\mathbf{H}_{\mathcal{B}(\mathbf{u}, L)}^{(N)}$. The following definitions are modifications of Definitions 2.1 and 3.1.

Definition 4.1. Given $E \in \mathbb{R}$ and $m^* \geq 1$, an N -particle ball $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L)$ is called (E, m_N) -nonsingular $((E, m_N)\text{-NS})$, if $\forall \mathbf{y} \in \partial^- \mathcal{B}$,

$$C_{\mathcal{Z}}^{2N} L^{Nd} \cdot \left| G_{\mathcal{B}}^{(N)}(\mathbf{x}, \mathbf{y}; E) \right| \leq e^{-\gamma(m_N, L)L}, \quad (4.3)$$

where

$$\gamma(m_N, L) := m_N(1 + L^{-1/8}). \quad (4.4)$$

Otherwise, \mathcal{B} is called (E, m_N) -singular $((E, m_N)\text{-S})$.

Definition 4.2. Given $E \in \mathbb{R}$, $\tau > 0$, $m^* \geq 1$ and integers $k, K \geq 0$, we say that a ball $\mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})$ is (E, m_N, K, τ) -good $((E, m_N, K, \tau)\text{-G})$ if $\mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})$ is (E, β) -CNR (cf. Definition 2.4) and contains no collection of $\geq K+1$ pairwise L_k^τ -distant $(E, m_N)\text{-S}$ balls of radius L_k .

We will assume that L_0 is large enough so that $L_k^\tau > 8NL_k$ for all $k \geq 0$ (cf. lemma 2.7).

These definitions are adopted throughout the current section.

A pre-requisite for the proof of the following statement is Appendix B.

Lemma 4.1. (Cf. Lemma 3.1.) *If $\mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})$ is (E, m_N, K, τ) -G and L_0 is large enough, then $\mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})$ is (E, m_N) -NS.*

Proof. Set $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})$ and fix $\mathbf{y} \in \partial^- \mathcal{B}$. Set

$$f_{\mathbf{y}} : \mathbf{z} \mapsto \left| G_{\mathcal{B}}^{(N)}(\mathbf{z}, \mathbf{y}; E) \right|.$$

The assumptions of Lemma 4.1 imply that \exists a (possibly empty) collection of $K' \leq K$ balls $\mathcal{B}(\mathbf{u}_j, 2L_k^\tau) \subset \mathcal{B}$ such that any ball $\mathcal{B}(\mathbf{v}, L_k) \subset \mathcal{B} \setminus \bigcup_{j=1}^{K'} \mathcal{B}(\mathbf{u}_j, 4NL_k)$ is (E, m_N) -NS. With $\mathcal{L}_r(\mathbf{u})$ standing, as before, for a spherical layer $\{\mathbf{z} \in \mathcal{Z} : \rho(\mathbf{z}, \mathbf{u}) = r\}$, we set:

$$\Xi := \left\{ \mathbf{x} \in \mathcal{B}_{L_{k+1}-5L_k-1}(\mathbf{u}) : \mathcal{L}_{d(\mathbf{u}, \mathbf{x})}(\mathbf{u}) \cap \bigcup_{j=1}^{K'} \mathcal{B}_{4NL_k}(\mathbf{u}_j) \neq \emptyset \right\}.$$

Then any ball $\mathcal{B}(\mathbf{v}, L_k) \subset \mathcal{B}$ with $\mathbf{v} \in \mathcal{B} \setminus \Xi$ is (E, m_N) -NS. Also, define the function $R = R_{f, \Xi}$ as in Eqn (B.3).

Owing to the assumptions of the lemma, the set Ξ is covered by a family of annuli with common center \mathbf{u} and total width $\leq 4KNL_k$. By Lemma B.2 $f_{\mathbf{y}}$ is (L_k, q, Ξ) -dominated in \mathcal{B} . Here (cf. (B.8) with $\delta = 1$)

$$\begin{aligned} -\ln q &= m_N(1 + L_k^{-1/8})L_k - L_{k+1}^\beta - \ln(C_{\mathcal{Z}}^N L_{k+1}^{Nd}) \\ &\geq m_N L_k + (m_N L_k^{7/8} - 2L_k^{\alpha\beta}) \geq L_k m_N \left(1 + \frac{1}{2} L_k^{-1/8}\right), \end{aligned}$$

where the last inequality follows from the form of m_N in (4.2). By virtue of Lemma B.1 (cf. Eqn (B.5)),

$$f_{\mathbf{y}}(\mathbf{u}) \leq q^{(L_{k+1}-2K(L_k^\tau+L_k))/(1+L_k)} M(f, \mathcal{B}).$$

One can see that $-\ln f_{\mathbf{y}}(\mathbf{u})$ is

$$\begin{aligned} &\geq -\ln \left\{ \left(e^{-m_N(1+L_k^{-1/8}/2)L_k} \right)^{(L_{k+1}-4KL_k^\tau)/(1+L_k)} e^{L_{k+1}^\beta} \right\} \\ &= m_N \left(1 + L_k^{-1/8}/2\right) L_{k+1} \cdot \frac{1 - (4NK+1)L_{k+1}^{-1+\frac{\tau}{\alpha}}}{1 + L_k^{-1}} - L_{k+1}^{1/4} \end{aligned}$$

which can be made

$$\geq L_{k+1} m_N \left(1 + \frac{1}{4} L_k^{-1/8} - L_{k+1}^{-3/4}\right) \geq \gamma(m_N, L_{k+1}) L_{k+1} + \ln \sharp(\partial \mathcal{B}),$$

assuming L_0 is large enough. This leads to the assertion of Lemma 4.1. \square

4.2 Localization in WI balls

The main result of Section 4.2 is Theorem 4.3. We begin with an analog of Lemma 3.3.

Lemma 4.2. *Fix $E \in \mathbb{R}$ and consider a WI ball $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L_k)$ with a canonical factorization $\mathcal{B} = \mathcal{B}' \times \mathcal{B}''$ and with reduced Hamiltonians $\mathbf{H}' = \mathbf{H}_{\mathcal{B}'}^{(N')}$ and $\mathbf{H}'' = \mathbf{H}_{\mathcal{B}''}^{(N'')}$ (cf. (2.25)). Assume \mathcal{B} is (E, β) -NR. Suppose in addition that $\forall \lambda'' \in \Sigma(\mathbf{H}'')$ the N' -particle ball \mathcal{B}' is $(E - \lambda'', m_{N'})$ -NS and $\forall \lambda' \in \Sigma(\mathbf{H}')$ the N'' -particle ball \mathcal{B}'' is $(E - \lambda', m_{N''})$ -NS. Then ball \mathcal{B} is (E, ν_N) -S.*

Proof. See Section A.2. □

Consider the following property (replacing $\mathcal{S}(N, k)$; cf. Eqn (3.5)).

$$\mathcal{S}_{\text{EXP}}(N, k): \quad \forall E \in \mathbb{R}, \quad 1 \leq n \leq N \text{ and } \mathbf{u} \in \mathcal{Z}^n$$

$$\mathbb{P} \left\{ \text{ball } \mathcal{B}^{(n)}(\mathbf{u}, L_k) \text{ is } (E, m_n)\text{-S} \right\} \leq L_k^{-P(n)}. \quad (4.5)$$

Theorem 4.3. *Suppose that property $\mathcal{S}_{\text{EXP}}(N - 1, k)$ holds for some given L_0, α, β, τ and $m^*, P^* \geq 1$. Take L_0 large enough. Then $\forall k \geq 0$ the following holds true. Assume that $\mathcal{S}_{\text{EXP}}(N - 1, k)$ holds. Then $\forall E$ and a WI ball $\mathcal{B}^{(N)}(\mathbf{u}, L_k)$,*

$$\mathbb{P} \left\{ \mathcal{B}^{(N)}(\mathbf{u}, L_k) \text{ is } (E, m_N)\text{-S} \right\} \leq L_{k+1}^{-\frac{3}{2}P(N)}. \quad (4.6)$$

Consequently, if L_0 is large enough then $\forall E$ and ball $\mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})$,

$$\begin{aligned} \mathbb{P} \left\{ \mathcal{B}^{(N)}(\mathbf{u}, L_{k+1}) \text{ contains a WI } (E, m_N)\text{-S ball of radius } L_k \right\} \\ \leq C_{\mathcal{Z}}^N L_{k+1}^{Nd} \cdot L_{k+1}^{-\frac{3}{2}P(N)} \leq \frac{1}{4} L_{k+1}^{-\frac{5}{4}P(N)}. \end{aligned} \quad (4.7)$$

Proof. First, we prove the bound in (4.6). As in the proof of Theorem 3.4, set $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L_k)$ and consider the canonical factorization $\mathcal{B} = \mathcal{B}' \times \mathcal{B}''$, with reduced Hamiltonians \mathbf{H}' and \mathbf{H}'' . Given $E \in I$, introduce the event $\mathcal{S} = \mathcal{S}(E, N)$:

$$\mathcal{S} = \{\omega : \mathcal{B} \text{ is WI and } (E, m_N)\text{-S}\}.$$

We have the following elementary inequality:

$$\begin{aligned} \mathbb{P} \{ \mathcal{S} \} &< \mathbb{P} \{ \mathcal{B} \text{ is not } E\text{-FNR} \} \\ &+ \mathbb{P} \{ \mathcal{B} \text{ is } E\text{-FNR and } (E, m_N)\text{-S} \}. \end{aligned} \quad (4.8)$$

As earlier, the first term in the RHS of (4.8) is assessed in Theorem 2.8, so we focus on the second summand. Apply Lemma 3.3 and introduce events \mathcal{S}' and \mathcal{S}'' by following the framework of Eqn (2.26) and (2.27). Then, with $m' = m_{N'}$,

$$\mathbb{P} \{ \mathcal{S}' \} = \mathbb{E} \left[\mathbb{P} \left\{ \exists \lambda'' \in \Sigma(\mathbf{H}'') : \mathcal{B}' \text{ is } (E - \lambda'', m')\text{-S} \mid \mathfrak{F}'' \right\} \right].$$

By definition of the canonical decomposition, $\Pi \mathcal{B}' \cap \Pi \mathcal{B}'' = \emptyset$, and since the random field V is IID, for any $E'' \in \mathbb{R}$, including $E - \lambda''$, the conditional probability does not depend on the condition:

$$\mathbb{P} \{ \mathcal{B}' \text{ is } (E'', m')\text{-S} \mid \mathfrak{F}'' \} \stackrel{\text{a.s.}}{=} \mathbb{P} \{ \mathcal{B}' \text{ is } (E'', m')\text{-S} \}. \quad (4.9)$$

On the other hand, by the assumed property $S(N-1, k)$,

$$\mathbb{P} \{ \mathcal{B}' \text{ is } (E'', m')\text{-S} \} \leq C_{\mathcal{Z}}^{-2N} L_k^{-P(N-1)} = C_{\mathcal{Z}}^{-2N} L_k^{-2P(N)}. \quad (4.10)$$

Therefore, in analogy with (2.27), we obtain that

$$\mathbb{P} \{ \mathcal{S}' \} \leq \# \mathcal{B}'' \sup_{E'' \in \mathbb{R}} \mathbb{P} \{ \mathcal{B}' \text{ is } (E'', m_N)\text{-S} \} \leq C_{\mathcal{Z}}^N L_k^{Nd} L_k^{-P(N-1)}; \quad (4.11)$$

after the substitution $P(N-1) = 2\alpha P(N)$ (cf. (4.2)), the RHS can be made $\leq C_{\mathcal{Z}}^N \frac{1}{2} L_{k+1}^{-\frac{3}{2}P(N)}$, provided $P(N) > 2Nd\alpha^{-1}$. The latter inequality follows from the bound in (4.2).

Summarising this calculation, we obtain

$$\mathbb{P} \{ \exists \lambda'' \in \Sigma'' : \mathcal{B}' \text{ is } (E - \lambda'', m')\text{-S} \} \leq \frac{1}{2} L_{k+1}^{-\frac{3}{2}P(N)}. \quad (4.12)$$

Similarly, with $m'' = m_{N''}$,

$$\mathbb{P} \{ \exists \lambda' \in \Sigma' : \mathcal{B}'' \text{ is } (E - \lambda', m'')\text{-S} \} \leq \frac{1}{2} L_{k+1}^{-\frac{3}{2}P(N)}. \quad (4.13)$$

Collecting (2.4), (4.8), (4.12), (3.11) and (4.13), the assertion (4.6) follows.

To prove (4.7), notice that the number of WI balls of radius L_k inside $\mathcal{B}^{(N)}(\mathbf{x}, L_{k+1})$ is bounded by the cardinality $\# \mathcal{B}^{(N)}(\mathbf{x}, L_{k+1})$, and the probability that a given WI ball is (E, m_N) -S satisfies (4.6). Therefore, the probability in the LHS of (4.7) is upper-bounded, for L_0 large enough, by

$$C_{\mathcal{Z}}^N L_{k+1}^{Nd} L_{k+1}^{-\frac{3}{2}P(N)} = L_{k+1}^{-\frac{5}{4}P(N)} \cdot C_{\mathcal{Z}}^N L_{k+1}^{-\frac{1}{4}P(N)+Nd} \leq \frac{1}{4} L_{k+1}^{-\frac{5}{4}P(N)}$$

since $P(N) \geq P(N^*) > 4Nd$, by virtue of (4.2). \square

4.3 The probabilistic scaling step

As in Section 3.2, we introduce probabilities P_k , Q_{k+1} and S_{k+1} . The following statement is a direct analog of Lemma 3.1 for the scaling scheme (4.1).

Lemma 4.4. *Let us be given a positive integer L_0 and values $\alpha > 0$, $\tau = 1$, $\beta \in (0, 1)$ and $m^*, P^* \geq 1$. If a ball $\mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})$ is (E, m_N, K, τ) -G where $K = 1$, then it is (E, m_N) -NS. The assertion remains valid if the condition (E, β) -NR (figuring in the definition of the (E, β) -CNR property) is replaced by a weaker assumption:*

$$\text{dist} \left(\Sigma \left(\mathbf{H}_{\mathcal{B}^{(N)}(\mathbf{u}, L_{k+1})}^{(N)} \right), E \right) \geq e^{-L^\beta}. \quad (4.14)$$

Lemma 4.4 is a particular (and simpler) case of Lemma 4.2 in Ref. [18]. The latter has been adapted since then to various models and became a common place. Hence, we omit its proof (it is similar to that of Lemma 3.1).

Theorem 4.5 is an analog of Theorem 3.5.

Theorem 4.5. *Suppose that, for some given $\alpha > 1$, τ as in (4.2), $\beta \in (0, 1)$, and $m^*, P^* \geq 1$, property $S_{\text{EXP}}(N, 0)$ is satisfied with L_0 large enough. Then $S_{\text{EXP}}(N, k)$ holds true $\forall k \geq 0$ with the same L_0, β, τ, m^* and P^* .*

Proof. It suffices to derive $S_{\text{EXP}}(N, k+1)$ from $S_{\text{EXP}}(N, k)$, so assume the latter. By virtue of Lemma 3.1, if a ball $\mathcal{B}(\mathbf{u}, L_{k+1})$ is (E, β) -S, then

- either $\mathcal{B}(\mathbf{u}, L_{k+1})$ is (E, β) -R (with probability $\leq \frac{1}{4}Q_{k+1}$),
- or $\mathcal{B}(\mathbf{u}, L_{k+1})$ contains at least one WI (E, m_N) -NS ball of radius L_k ,
- or $\mathcal{B}(\mathbf{u}, L_{k+1})$ contains at least two SI and (E, m_N) -S balls of radius L_k .

By Eqn (4.7), the probability to have at least one WI (E, m_N) -S ball of radius L_k inside $\mathcal{B}(\mathbf{u}, L_{k+1})$ obeys $S_{k+1} \leq \frac{1}{4}L_{k+1}^{-P(N)}$.

Therefore, it remains to assess the probability to have a collection of at least two SI and $(E, 1, m_N)$ -S balls of radius L_k inside $\mathcal{B}(\mathbf{u}, L_{k+1})$. The number of such collections is $\leq C_{\mathcal{Z}}^{2N} L_{k+1}^{(2Nd)}$, thus

$$P_{k+1} \leq \frac{1}{2}C_{\mathcal{Z}}^{2N} L_{k+1}^{2Nd} P_k^2 + S_{k+1} + Q_{k+1}.$$

By Theorem 2.1, $Q_{k+1} \leq C_W L_k^{(N+1)d} e^{-L_{k+1}^\beta}$ where $\beta > 0$, thus for L_0 large enough, $Q_k \leq \frac{1}{4}L_{k+1}^{-P(N)}$ for any $k \geq 0$. Thus we can write

$$P_{k+1} \leq \frac{1}{2}C_{\mathcal{Z}}^{2N} L_{k+1}^{2Nd} P_k^2 + \frac{1}{4}L_{k+1}^{-P(N)} + \frac{1}{4}L_{k+1}^{-P(N)}, \quad (4.15)$$

and the RHS can be made $< L_{k+1}^{-P(N)}$, whenever $P(N) > 4Nd$ and L_0 is large enough. Again, the condition $P(N) > 4Nd$ follows from (4.2). \square

4.4 Conclusion: exponential decay of eigenfunctions

In this section, as before, the condition (V), as well as the property (RCM) stemming from it (cf. Theorem 2.3), is always assumed, so we do not repeat it in the formulations of theorems 4.6 and 4.7.

Recall that under the assumption (V), the spectrum of the Hamiltonian $\mathbf{H}(\omega)$, as well as the spectra of its restrictions to arbitrary finite balls, is a.s. bounded by a value $O(|g|, N, d)$, so we can restrict our analysis to a compact energy interval $I_g^* = I_g^*(N, d) \subset \mathbb{R}$ of length $|I_g^*|$. Below we assume that such an interval is fixed.

An analog of Theorem 3.6 is the following

Theorem 4.6. *Suppose we are given two $3NL$ -distant balls $\mathcal{B}_L(\mathbf{x})$, $\mathcal{B}_L(\mathbf{y})$ and numbers $a_L, q_L > 0$ such that for any $E \in \mathbb{R}$*

$$\max \left[\mathbb{P} \{ \mathbf{F}_{\mathbf{x}}(E) > a_L \}, \mathbb{P} \{ \mathbf{F}_{\mathbf{y}}(E) > a_L \} \right] \leq q_L.$$

Then for any $b > 0$, one has

$$\mathbb{P} \{ \exists E \in I_g^* : \min(\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)) \geq a_L \} \leq 2|I_g^*|b^{-1}q_L + \tilde{h}_L(4b), \quad (4.16)$$

where

$$\tilde{h}(s) = C\overline{K}^2 L^A s^B + C' L^{A'} s^{B'}, \quad \overline{K} = \max\{\#\mathcal{B}_L(\mathbf{x}), \#\mathcal{B}_L(\mathbf{y})\}. \quad (4.17)$$

The reason why we need a separate bound (4.16) is that the derivation of the variable-energy estimates based on Theorem 3.6 gives rise to exponential decay of eigenfunctions only if the probabilistic bounds obtained in the fixed-energy analysis in the balls of size L are also exponential in L ; this can be seen in the condition (3.16).

In the proof given below, we will use the following auxiliary result.

Theorem 4.7 (Cf. [12, Theorem 4]). *Suppose a ball $\mathcal{B}_L(\mathbf{x})$ and numbers $a_L, q_L > 0$ are such that for all $E \in \mathbb{R}$*

$$\mathbb{P}\{\mathbf{F}_{\mathbf{x}}(E) > a_L\} \leq q_L. \quad (4.18)$$

Set $K = \sharp \mathcal{B}_L(\mathbf{x})$. Then the following properties (A), (B) hold true:

(A) *For any $b > q_L$ there exists an event \mathcal{S}_b with $\mathbb{P}\{\mathcal{S}_b\} \leq b^{-1}q_L$ and such that for any $\omega \notin \mathcal{S}_b$, the set of energies*

$$\mathcal{E}_{\mathbf{x}}(a_L) = \mathcal{E}_{\mathbf{x}}(a_L; \omega) := \{\mathbf{F}_{\mathbf{x}}(E) \geq a_L\}$$

is covered by $K' < 3K$ intervals $J_i = [E_i^-, E_i^+]$, of total length $\sum_i |J_i| \leq b$.

(B) *Consider the parametric operator family $\mathbf{A}(t) = \mathbf{H}_{\mathcal{B}} + t\mathbf{1}$, $t \in \mathbb{R}$. The endpoints $E_i^{\pm}(t)$ for the operators $\mathbf{A}(t)$ (replacing $\mathbf{H}_{\mathcal{B}_L(\mathbf{x})}$) have the form*

$$E_i^{\pm}(t) = E_i^{\pm} + t, \quad t \in \mathbb{R}.$$

Proof of Theorem 4.7. (A) Set for brevity $\mathcal{B} = \mathcal{B}_L(\mathbf{x})$. We have that

$$\mathbf{F}_{\mathbf{x}} = \max_{\mathbf{y} \in \partial^{-}\mathcal{B}} |\mathbf{F}_{\mathbf{x}, \mathbf{y}}|, \text{ where } \mathbf{F}_{\mathbf{x}, \mathbf{y}} := G_{\mathcal{B}}(\mathbf{x}, \mathbf{y}; E).$$

Fix \mathbf{y} and consider $\mathbf{F}_{\mathbf{x}, \mathbf{y}}$ as a rational function

$$\mathbf{F}_{\mathbf{x}, \mathbf{y}} : E \mapsto \sum_{k=1}^K \frac{c_k}{E_k - E} = \sum_{k=1}^K \frac{\langle \mathbf{1}_{\mathbf{x}} | \psi_k \rangle \langle \psi_k | \mathbf{1}_{\mathbf{y}} \rangle}{E_k - E}. \quad (4.19)$$

Its derivative is a ratio of two polynomials:

$$\frac{d}{dE} \mathbf{F}_{\mathbf{x}, \mathbf{y}}(E) = - \sum_k c_k (E_k - E)^{-2} =: \mathcal{P}(E) / \mathcal{Q}(E),$$

with $\deg \mathcal{P} \leq 2K - 2$. Hence, it has $\leq 2K - 2$ zeros and $\leq K$ poles, so $\mathbf{F}_{\mathbf{x}, \mathbf{y}}$ has $< 3K$ intervals of monotonicity. Then the total number of monotonicity intervals for all functions $\mathbf{F}_{\mathbf{x}, \mathbf{y}}$ is upper-bounded by $(\sharp \partial^{-}\mathcal{B}_L(\mathbf{u})) \cdot 3K \leq 3K^2$. Admitting the value $+\infty$ for the functions $|\mathbf{F}_{\mathbf{x}, \mathbf{y}}|$, we can write

$$\cup_{\mathbf{y}} \{E : |\mathbf{F}_{\mathbf{x}, \mathbf{y}}(E)| \geq a\} = \cup_{k=1}^K J_k = [E_k^-, E_k^+] \subset I_g^*.$$

Let $\mathcal{S}_{b, \mathbf{x}} = \{\omega : \text{mes}\{E \in I_g^* : \mathbf{F}_{\mathbf{x}}(E) \geq a\} \geq b\}$. By the Chebychev inequality combined with the Fubini theorem, we have

$$\begin{aligned} \mathbb{P}\{\mathcal{S}_{b, \mathbf{x}}\} &\leq b^{-1} \mathbb{E}[\mathcal{S}_{b, \mathbf{x}}] = b^{-1} \mathbb{E} \left[\int_{I_g^*} \mathbf{1}_{\{\mathbf{F}_{\mathbf{x}}(E) \geq a\}} dE \right] \\ &= b^{-1} \int_{I_g^*} \mathbb{E}[\mathbf{1}_{\{\mathbf{F}_{\mathbf{x}}(E) \geq a\}}] dE = b^{-1} \int_{I_g^*} \mathbb{P}\{\mathbf{F}_{\mathbf{x}}(E) \geq a\} dE \\ &\leq b^{-1} |I_g^*| q_L. \end{aligned} \quad (4.20)$$

So, for all $\omega \notin \mathcal{S}_{b, \mathbf{x}}$, $\sum_k |J_k| \leq \text{mes}\{E \in I_g^* : \mathbf{F}_{\mathbf{x}} \geq a\} \leq b$. This yields property (A).

(B) The operators $\mathbf{A}(t)$ share common eigenvectors; the latter determine the coefficients c_k in (4.19), so we can choose the eigenfunctions $\psi_k(t)$ constant in t and obtain $c_k(t) \equiv c_k(0)$. The eigenvalues of $A(t)$ have the form $E_k(t) = E_k + t$. Therefore, $\mathbf{F}_{\mathbf{x}, \mathbf{y}}(E; t) = \mathbf{F}_{\mathbf{x}, \mathbf{y}}(E - t; 0)$, and $J_k(t) = [E_k^- + t, E_k^+ + t]$.

Proof of Theorem 4.6. Fix $b > 0$ and let $\mathcal{S}_{b,\mathbf{z}} = \{\omega : \text{mes}\{E : \mathbf{F}_{\mathbf{z}}(E) \geq a\} \geq b\}$ for $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}$, $\mathcal{S}_b = \mathcal{S}_{b,\mathbf{x}} \cup \mathcal{S}_{b,\mathbf{y}}$. Let \mathcal{S} be the event figuring in the LHS of (4.16). Using the bounds of the form (4.20) on $\mathbb{P}\{\mathcal{S}_{b,\mathbf{x}}\}$ and $\mathbb{P}\{\mathcal{S}_{b,\mathbf{y}}\}$, we have

$$\mathbb{P}\{\mathcal{S}\} \leq \mathbb{P}\{\mathcal{S}_b\} + \mathbb{P}\left\{\mathcal{S} \cap \mathcal{S}_b^c\right\} \leq 2|I_g^*|b^{-1}q_L + \mathbb{P}\left\{\mathcal{S} \cap \mathcal{S}_b^c\right\}.$$

It remains to assess $\mathbb{P}\left\{\mathcal{S} \cap \mathcal{S}_b^c\right\}$.

By Lemma 2.4, the $3NL$ -distant balls $\mathcal{B}_L(\mathbf{x})$, $\mathcal{B}_L(\mathbf{y})$ are weakly \mathcal{B} -separated for some $\mathcal{B} \subset \mathcal{Z}$. Consider the random variables $\xi = \xi_{\mathcal{B}} = \langle V(\cdot; \omega) \rangle_{\mathcal{B}}$, $\eta_z(\omega) = V(z; \omega) - \xi(\omega)$, $z \in \mathcal{B}$, and let $\mathfrak{F}_{\mathcal{B}}$ be the sigma-algebra generated by $\{\eta_z, z \in \mathcal{B}; V(u; \cdot), u \notin \mathcal{B}\}$. Introduce the continuity modulus $\mathfrak{s}_{\xi}(\cdot | \mathfrak{F}_{\mathcal{B}})$ of the conditional probability distribution function $F_{\xi}(t | \mathfrak{F}_{\mathcal{B}}) = \mathbb{P}\{\xi \leq t | \mathfrak{F}_{\mathcal{B}}\}$; it satisfies the condition (RCM) with some $C', A', B', C'', A'', B''$. The representation $V(z; \omega) = \xi(\omega) + \eta_z(\omega)$ for $z \in \mathcal{B}$ implies $\mathbf{H}_{\mathcal{B}} = n_1\xi(\omega) + \mathbf{A}(\omega)$, where \mathbf{A} is $\mathfrak{F}_{\mathcal{B}}$ -measurable.

For any $\omega \notin \mathcal{S}_b$, the energies E where $\mathbf{F}_{\mathbf{x}}(E) \geq a$ are covered by a union of intervals $J_{\mathbf{x},i}$ with $|J_{B\mathbf{x},i}| =: \epsilon_{\mathbf{x},i}$, $\sum_i \epsilon_{\mathbf{x},i} \leq 2b$. By assertion (B) of Theorem 4.7, we have

$$J_{\mathbf{x},i}(\omega) = [\lambda_{\mathbf{x},i}^-(\omega) + n_1\xi(\omega), \lambda_{\mathbf{x},i}^+(\omega) + n_1\xi(\omega)],$$

where $\lambda_{\mathbf{x},i}^{\pm}$ are $\mathfrak{F}_{\mathcal{B}}$ -measurable.

Similarly, introduce the intervals $J_{\mathbf{y},j}$ with $|J_{B\mathbf{y},j}| =: \epsilon_{\mathbf{y},j}$, $\sum_j \epsilon_{\mathbf{y},j} \leq 2b$, and

$$J_{\mathbf{y},j}(\omega) = [\lambda_{\mathbf{y},j}^-(\omega) + n_2\xi(\omega), \lambda_{\mathbf{y},j}^+(\omega) + n_2\xi(\omega)], \quad n_2 < n_1.$$

We have

$$\begin{aligned} \{\omega : J_{\mathbf{x},i} \cap J_{\mathbf{y},j} \neq \emptyset\} \cap \mathcal{S}_b^c &\subset \{\omega : |\lambda_{\mathbf{x},i} - \lambda_{\mathbf{y},j}| \leq \epsilon_{\mathbf{x},i} + \epsilon_{\mathbf{y},j}\} \cap \mathcal{S}_b^c \\ &\subset \{\omega : |(n_1 - n_2)\xi - \mu_{i,j}(\omega)| \leq 4b\}, \end{aligned}$$

with some $\mathfrak{F}_{\mathcal{B}}$ -measurable $\mu_{i,j}$. Let $n = n_1 - n_2 \geq 1$ (recall that $\mathbb{Z} \ni n_1 - n_2 > 0$). By (RCM),

$$\begin{aligned} \mathbb{P}\{|(n_1 - n_2)\xi - \mu_{i,j}| \leq 4b\} &\leq \mathbb{E}[\mathbb{P}\{|(n_1 - n_2)\xi - \mu_{i,j}| \leq 4b | \mathfrak{F}_{\mathcal{B}}\}] \\ &\leq \mathbb{P}\left\{\mathfrak{s}_{\xi}(4b | \mathfrak{F}_{\mathcal{B}}) \geq C' L^{A'} (4b)^{B'}\right\} + C' L^{A'} s^{B'}. \end{aligned}$$

Taking the sum over all i and j , we obtain the asserted bound. \square

Setting $L = L_k$, $k \geq 0$, and

$$a_{L_k} = e^{-m_N L_k}, \quad q_{L_k} = L_k^{-P(N)}, \quad b = L_k^{-P(N)/2},$$

we come to the following result, marking the end of the proof of our main theorem. Recall that the strong dynamical localization bounds have already been established, and we only need to prove exponential decay of the eigenfunctions.

Corollary 4.8. *For $k \geq 0$ and any pair of $3NL_k$ -distant balls $\mathcal{B}_{L_k}(\mathbf{x})$, $\mathcal{B}_{L_k}(\mathbf{y})$ the following bound holds true:*

$$\mathbb{P}\{\exists E \in \mathbb{R} : \mathcal{B}_{L_k}(\mathbf{x}) \text{ and } \mathcal{B}_{L_k}(\mathbf{y}) \text{ are } (E, m_N)\text{-S}\} \leq C L_k^{-P(N)/2}. \quad (4.21)$$

Consequently, for $|g|$ large enough, with probability one, the operator $\mathbf{H}_{\mathcal{V}}^{(N)}$, for any $\mathcal{V} \subseteq \mathcal{Z}^N$ such that $\mathcal{V} \supseteq \mathcal{B}_{L_k}(\mathbf{x}), \mathcal{B}_{L_k}(\mathbf{y})$, has a pure point spectrum, and its eigenfunctions obey (1.12).

Proof. The first assertion follows from Theorem 4.6. The second assertion is a well-known result going back to [18]. In fact, the proof of Lemma 3.1 from [18] can be adapted to pairs of balls $\mathcal{B}_{L_k}(\mathbf{x}), \mathcal{B}_{L_k}(\mathbf{y}) \subset \mathcal{Z}^N$ at distance $\geq CL_k$, with a constant $C \in (0, +\infty)$. The key fact is that structure of the random potential (single- or multi-particle) is irrelevant to the proof of [18, Lemma 3.1]. \square

Appendix A Proof of Lemmas 3.3 and 4.2

A.1 Proof of Lemma 3.3

Step 1. Approximate decoupling. In accordance with the canonical decomposition, write $\mathbf{u} = (\mathbf{u}', \mathbf{u}'')$ where $\mathbf{u}' = \mathbf{u}_{\mathcal{J}} \in \mathcal{Z}^{N'}$, $\mathbf{u}'' = \mathbf{u}_{\mathcal{J}^c} \in \mathcal{Z}^{N''}$. Let $\mathcal{B} = \mathcal{B}' \times \mathcal{B}''$ be the corresponding canonical factorization of the WI ball $\mathcal{B} = \mathcal{B}^{(N)}(\mathbf{u}, L_k)$ with $\mathcal{B}' = \mathcal{B}^{(N')}(\mathbf{u}', L_k)$, $\mathcal{B}'' = \mathcal{B}^{(N'')}(\mathbf{u}'', L_k)$. Cf. Eqns (2.15)–(2.16).

By Definition 2.3 and Lemma 2.6, the graph distance between projected configurations (in \mathcal{Z}) satisfies $d(\Pi_{\mathcal{J}}\mathbf{x}, \Pi_{\mathcal{J}^c}\mathbf{x}) > L_k$, yielding that $\forall \mathbf{x} \in \mathcal{B}$,

$$d(\Pi_{\mathcal{J}}\mathbf{x}, \Pi_{\mathcal{J}^c}\mathbf{x}) > L_k.$$

Consider representation (2.17):

$$\mathbf{H}_{\mathcal{B}}^{(N)} = \mathbf{H}_{\mathcal{B}}^{\text{ni}} + \mathbf{U}_{\mathcal{B}', \mathcal{B}''} \text{ where } \mathbf{H}_{\mathcal{B}}^{\text{ni}} = \mathbf{H}_{\mathcal{B}'}^{(N')} \otimes \mathbf{I}^{(N'')} + \mathbf{I}^{(N')} \otimes \mathbf{H}_{\mathcal{B}''}^{(N'')}. \quad (\text{A.1})$$

(The superscript "ni" stands for non-interacting.) Here $\mathbf{U}_{\mathcal{B}', \mathcal{B}''}$ is the operator of multiplication by the function

$$\mathbf{x} = (x_1, \dots, x_N) \in \mathcal{B} \mapsto \sum_{1 \leq i < j \leq N} \mathbf{1}(i \in \mathcal{J}, j \in \mathcal{J}^c) U(d(x_i, x_j)). \quad (\text{A.2})$$

According to assumption (U), the norm of operator $\mathbf{U}_{\mathcal{B}', \mathcal{B}''}$ obeys

$$\|\mathbf{U}_{\mathcal{B}', \mathcal{B}''}\| \leq C(N' \cdot N'')e^{-L_k^\zeta} \leq CN^2e^{-L_k^\zeta}, \quad (\text{A.3})$$

with $C = C_U$ as in (1.10).

The eigenvalues of $\mathbf{H}_{\mathcal{B}}^{\text{ni}}$ are the sums $E_{a,b} = \lambda_a + \mu_b$, where λ_a form the spectrum $\Sigma(\mathbf{H}_{\mathcal{B}'}^{(N')})$ and μ_b the spectrum $\Sigma(\mathbf{H}_{\mathcal{B}''}^{(N'')})$. The eigenvectors of $\mathbf{H}_{\mathcal{B}}^{\text{ni}}$ can be chosen in the form $\phi_a \otimes \psi_b$ where $\{\phi_a\}$ are eigenvectors of $\mathbf{H}_{\mathcal{B}'}^{(N')}$ and $\{\psi_b\}$ of $\mathbf{H}_{\mathcal{B}''}^{(N'')}$.

Step 2. Nonresonance properties. Next we infer from the assumed (E, β) -NR property of \mathcal{B} (with regard to the resolvent $\mathbf{G}_{\mathcal{B}}^{(N)}(E) = (\mathbf{H}_{\mathcal{B}}^{(N)} - E\mathbf{I})^{-1}$) a similar (albeit a weaker) property for the resolvent $\mathbf{G}^{\text{ni}}(E) = (\mathbf{H}_{\mathcal{B}}^{\text{ni}} - E\mathbf{I})^{-1}$. By the min-max principle,

$$\begin{aligned} \text{dist}\left(\Sigma(\mathbf{H}_{\mathcal{B}}^{\text{ni}}), E\right) &\geq \text{dist}\left(\Sigma\left(\mathbf{H}_{\mathcal{B}}^{(N)}\right), E\right) - \|\mathbf{U}_{\mathcal{B}', \mathcal{B}''}\| \\ &\geq 2e^{-L_k^\beta} - C_U e^{-4L_k^\zeta} \geq e^{-L_k^\beta}, \end{aligned} \quad (\text{A.4})$$

provided that $\beta < \zeta$ (which is one of conditions (3.1)) and L_0 is large enough.

For each pair (λ_a, μ_b) , the non-resonance condition $|E - (\lambda_a + \mu_b)| \geq e^{L_k^\beta}$ reads as $|(E - \lambda_a) - \mu_b| \geq e^{L_k^\beta}$ and also as $|(E - \mu_b) - \lambda_a| \geq e^{L_k^\beta}$. In terms of resolvents $\mathbf{G}_{\mathcal{B}}^{(N)}(E)$ and $\mathbf{G}_{\mathcal{B}}^{\text{ni}}(E)$ we then have:

$$\left\| \mathbf{G}_{\mathcal{B}}^{(N)}(E) \right\| \leq \frac{1}{2} e^{L_k^\beta} < e^{L_k^\beta}, \quad \left\| \mathbf{G}_{\mathcal{B}}^{\text{ni}}(E) \right\| \leq e^{L_k^\beta}. \quad (\text{A.5})$$

Step 3. Analytic perturbation estimates. We begin with analyzing the resolvent $\mathbf{G}_{\mathcal{B}}^{\text{ni}}(E)$. Start with the identities for the GF $G_{\mathcal{B}}^{\text{ni}}(\mathbf{u}, \mathbf{y}; E)$:

$$G_{\mathcal{B}}^{\text{ni}}(\mathbf{u}, \mathbf{y}; E) = \sum_{\lambda_a} \sum_{\mu_b} \frac{\phi_a(\mathbf{u}') \phi_a(\mathbf{y}') \psi_b(\mathbf{u}'') \psi_b(\mathbf{y}'')}{(\lambda_a + \mu_b) - E} \quad (\text{A.6})$$

$$= \sum_{\lambda_a} \phi_a(\mathbf{u}') \phi_a(\mathbf{y}') G_{\mathcal{B}''}^{(N'')}(\mathbf{u}'', \mathbf{y}''; E - \lambda_a) \quad (\text{A.7})$$

$$= \sum_{\mu_b} \psi_b(\mathbf{u}'') \psi_b(\mathbf{y}'') G_{\mathcal{B}'}^{(N')}(\mathbf{u}', \mathbf{y}'; E - \mu_b). \quad (\text{A.8})$$

By assumptions of the lemma,

$$\begin{aligned} & \bullet \forall \mu_b \in \Sigma \left(\mathbf{H}_{\mathcal{B}''}^{(N'')} \right), \text{ the ball } \mathcal{B}' \text{ is } (\mu_b, \delta, \nu_{N'})\text{-NS}, \\ & \bullet \forall \lambda_a \in \Sigma \left(\mathbf{H}_{\mathcal{B}'}^{(N')} \right), \text{ the ball } \mathcal{B}'' \text{ is } (\lambda_a, \delta, \nu_{N''})\text{-NS}. \end{aligned} \quad (\text{A.9})$$

For any $\mathbf{y} \in \partial^- \mathcal{B}$, either $\rho^{(N')}(\mathbf{u}', \mathbf{y}') = L_k$ or $\rho^{(N'')}(\mathbf{u}'', \mathbf{y}'') = L_k$. In the first case we infer from (A.8), combined with $(\mu_b, \delta, \nu_{N'})$ -NS property of ball \mathcal{B}' , that

$$|G_{\mathcal{B}}^{\text{ni}}(\mathbf{u}, \mathbf{y}; E)| \leq \sharp \mathcal{B}'' e^{-m_{N'} L_k^\delta + 2L_k^\beta}. \quad (\text{A.10})$$

Similarly, in the second case we obtain that

$$|G_{\mathcal{B}}^{\text{ni}}(\mathbf{u}, \mathbf{y}; E)| \leq \sharp \mathcal{B}' e^{-m_{N''} L_k^\delta + 2L_k^\beta}. \quad (\text{A.11})$$

In either case, the LHS is bounded by (cf. (2.20))

$$C_{\mathcal{Z}}^N L_k^{Nd} e^{-m_{N-1} L_k^\delta + 2L_k^\beta} \leq e^{-m_N L_k^\delta - L_0^\beta} \leq \frac{1}{2} e^{-m_N L_k^\delta}, \quad (\text{A.12})$$

provided that L_0 is large enough.

Now, to assess $G_{\mathcal{B}}^{(N)}(\mathbf{u}, \mathbf{y}; E)$, we use the second resolvent equation and write:

$$\begin{aligned} \left\| \mathbf{G}_{\mathcal{B}}^{(N)}(E) - \mathbf{G}_{\mathcal{B}}^{\text{ni}}(E) \right\| & \leq \left\| \mathbf{G}_{\mathcal{B}}^{\text{ni}}(E) \right\| \left\| \mathbf{U}_{\mathcal{B}', \mathcal{B}''} \right\| \left\| \mathbf{G}_{\mathcal{B}}^{(N)}(E) \right\| \\ & \leq C_U e^{2L_k^\beta - L_k^\zeta} \leq e^{-\frac{1}{2} L_k^\zeta} \leq \frac{1}{2} e^{-\nu_N L_k^\delta}, \end{aligned} \quad (\text{A.13})$$

provided that $\beta < \delta < \zeta$ and L_0 large enough (this holds in accordance with (3.1)).

Collecting (A.7), (A.8), (A.12) and (A.13), we get

$$\max_{\mathbf{y} \in \partial^- \mathcal{B}} \left| G_{\mathcal{B}}^{(N)}(\mathbf{u}, \mathbf{y}; E) \right| \leq \frac{1}{2} e^{-\nu_N L_k^\delta} + \frac{1}{2} e^{-\nu_N L_k^\delta} = e^{-\nu_N L_k^\delta}. \quad (\text{A.14})$$

We see that ball \mathcal{B} is (E, δ, ν_N) -NS.

A.2 Proof of Lemma 4.2

The line of the argument here follows, *mutatis mutandis*, that from the proof of Lemma 3.3.

Step 1. Approximate decoupling. We start as in the previous section, but have to achieve an exponential bound upon the GFs. The bound on the interaction (A.3) is to be modified accordingly:

$$\rho(\Pi_{\mathcal{J}}\mathcal{B}, \Pi_{\mathcal{J}^c}\mathcal{B}) \geq L_k^\tau \quad \text{and} \quad \|\mathbf{U}_{\mathcal{B}', \mathcal{B}''}\| \leq C_U N^2 e^{-ML_k}. \quad (\text{A.15})$$

Here M can be chosen arbitrarily large, provided that L_0 is large enough. Specifically, we require that $M \geq \max\{1, m_1\}$, hence $M \geq \max\{1, m_N\}$ for $1 \leq N \leq N^*$. Cf. (4.2).

The definitions of the operators $\mathbf{H}_{\mathcal{B}}^{\text{ni}}$ and $\mathbf{U}_{\mathcal{B}', \mathcal{B}''}$ (see (A.1) (A.2)) remain in force.

Step 2. Nonresonance properties. A direct analog of (A.4) is

$$\text{dist}[\Sigma(\mathbf{H}_{\mathcal{B}}^{\text{ni}}), E] \geq 2e^{-L_k^\beta} - e^{-ML_k} \geq e^{-L_k^\beta}; \quad (\text{A.16})$$

it implies, as before, that

$$\|\mathbf{G}_{\mathcal{B}}^{(N)}(E)\| \leq \frac{1}{2}e^{L_k^\beta} < e^{L_k^\beta}, \quad \|\mathbf{G}_{\mathcal{B}}^{\text{ni}}(E)\| \leq e^{L_k^\beta}. \quad (\text{A.17})$$

Step 3. Analytic perturbation estimates. We can use identities (A.6)–(A.8) and the (assumed) properties (A.9). Now, the estimates (A.10)–(A.10) are to be modified as follows.

Given $\mathbf{y} \in \partial^-\mathcal{B}$, we again have two possibilities. (i) $\rho^{(N')}(\mathbf{u}', \mathbf{y}') = L_k$, in which case we deduce from (A.8), combined with $(\mu_b, m_{N'})$ -NS property of the ball \mathcal{B}' , that

$$|G^{\text{ni}}(\mathbf{u}, \mathbf{y}; E)| \leq \sharp \mathcal{B}'' e^{-m_{n'} L_k + 2L_k^\beta}. \quad (\text{A.18})$$

The other case is where (ii) $\rho^{(N'')}(\mathbf{u}'', \mathbf{y}'') = L_k$ – then, similarly to (A.7), we have that

$$|G_{\mathcal{B}}^{\text{ni}}(\mathbf{u}, \mathbf{y}; E)| \leq \sharp \mathcal{B}' e^{-m_{n''} L_k + 2L_k^\beta}. \quad (\text{A.19})$$

In either case, with $C_{\mathcal{Z}}^N L_k^{Nd} \leq e^{L_k^\beta}$, the LHS of Eqn (A.7) is bounded by

$$C_{\mathcal{Z}}^N L_k^{Nd} e^{-m_{N-1} L_k + 2L_k^\beta} \leq e^{-m_N L_k - L_0^\beta} \leq \frac{1}{2} e^{-m_N L_k}. \quad (\text{A.20})$$

Now, by virtue of the second resolvent identity we have

$$\|\mathbf{G}_{\mathcal{B}}^{(N)}(E) - \mathbf{G}_{\mathcal{B}}^{\text{ni}}(E)\| \leq \|\mathbf{G}_{\mathcal{B}}^{\text{ni}}(E)\| \|\mathbf{U}_{\mathcal{J}, \mathcal{J}^c}\| \|\mathbf{G}_{\mathcal{B}}^{(N)}(E)\| \leq \frac{1}{2} e^{-m_N L_k}, \quad (\text{A.21})$$

since $M \geq 1$, $L_0 > 1$, $\beta \leq 1$. Collecting (A.7)–(A.8) and the bounds (A.20)–(A.21), we obtain

$$\max_{\mathbf{y} \in \partial^-\mathcal{B}} |G_{\mathcal{B}}^{(N)}(\mathbf{u}, \mathbf{y}; E)| \leq \frac{1}{2} e^{-m_N L_k} + e^{-2ML_k} \leq e^{-m_N L_k}.$$

Therefore, ball \mathcal{B} is (E, m_N) -NS.

Appendix B Dominated decay of functions on \mathcal{Z}

In this section we establish Lemmas B.1 and B.2 applicable to arbitrary locally finite, connected graphs, including $\mathcal{Z} = \mathcal{Z}^N$, $N \geq 2$. These lemmas are related to iterations of the GRI (see Eqn (2.1)) and provide an ingredient in the proof of Lemma 3.1 and 4.1. The argument here stems from [18], Lemma 4.2; in the case where $\mathcal{Z} = \mathbb{Z}^d$, it was presented in [17], Sect 2.6.

Definition B.1. (Cf. Definition 2.6.1 in [17]) *Let us be given a finite subset $\mathcal{V} \subset \mathcal{Z}$, a non-negative function $f : \mathcal{V} \rightarrow [0, \infty)$, a number $q \in (0, 1)$ and two integers $L \geq \ell \geq 1$. Take an N -particle ball $\mathcal{B}(\mathbf{u}, L) \subset \mathcal{V}$.*

- (1) *A point $\mathbf{x} \in \mathcal{B}^{(N)}(\mathbf{u}, L - \ell)$ is called (ℓ, q) -regular for the function f , if*

$$f(\mathbf{x}) \leq q M(f, \mathcal{B}(\mathbf{x}, \ell + 1)), \quad (\text{B.1})$$

and (ℓ, q) -singular, otherwise. Here and below, we set:

$$M(f, \mathcal{W}) = \sup [f(\mathbf{y}) : \mathbf{y} \in \mathcal{W}], \quad \mathcal{W} \subseteq \mathcal{V}. \quad (\text{B.2})$$

The set of all (ℓ, q) -regular points $\mathbf{x} \in \mathcal{B}(\mathbf{u}, L)$ for f is denoted by $\mathcal{R}_f(\mathbf{u}) = \mathcal{R}_{f,q,\ell}(\mathbf{u})$, and the set of all (ℓ, q) -singular points by $\mathcal{S}_f(\mathbf{u}) = \mathcal{S}_{f,q,\ell}(\mathbf{u})$.

- (2) *A spherical layer*

$$\mathcal{L}_r(\mathbf{u}) = \{\mathbf{y} \in \mathcal{Z} : d(\mathbf{u}, \mathbf{y}) = r\}$$

is called regular if $\mathcal{L}_r(\mathbf{u}) \subset \mathcal{R}_f(\mathbf{u})$.

- (3) *For $\mathbf{x} \in \mathcal{B}(\mathbf{u}, L - \ell)$, set*

$$\bar{r}(\mathbf{x}) := \begin{cases} \min [r \geq d(\mathbf{u}, \mathbf{x}) : \mathcal{L}_r(\mathbf{u}) \subset \mathcal{R}_f(\mathbf{u}) = \emptyset], \\ \text{if a regular layer } \mathcal{L}_r(\mathbf{u}) \text{ exists, with } r \geq d(\mathbf{u}, \mathbf{x}), \\ +\infty, \quad \text{if no such layer } \mathcal{L}_r(\mathbf{u}) \text{ exists,} \end{cases}$$

and

$$R_f(\mathbf{x}) (= R_{f,q,\ell}(\mathbf{x})) = \begin{cases} \bar{r}(\mathbf{x}) + \ell, & \bar{r}(\mathbf{x}) < +\infty, \\ +\infty, & \text{otherwise.} \end{cases} \quad (\text{B.3})$$

- (4) *Given a set $\Xi \subseteq \mathcal{V}$, function f is called (ℓ, q, Ξ) -dominated in $\mathcal{B}(L, \mathbf{u})$ if $\mathcal{S}_f(\mathbf{u}) \subset \Xi$, and for any $\mathbf{x} \in \mathcal{B}(\mathbf{u}, L - \ell)$ with $R_f(\mathbf{x}) < +\infty$, one has*

$$f(\mathbf{x}) \leq q M(f, \mathcal{B}(\mathbf{u}, R_f(\mathbf{x}))). \quad (\text{B.4})$$

Lemma B.1 (Cf. Theorem 2.6.1 in [17]). *Let function $f : \mathcal{V} \rightarrow \mathbb{R}_+$ be (ℓ, q, Ξ) -dominated in an N -particle ball $\mathcal{B}(\mathbf{u}, L)$, where $L \geq \ell \geq 0$. Assume that set Ξ is covered by a union \mathcal{U} of concentric annuli $\mathcal{B}(\mathbf{u}, b_j) \setminus \mathcal{B}(\mathbf{u}, a_j - 1)$, with*

$$w(\mathcal{U}) := \sum_j (b_j - a_j + 1) \leq L - \ell.$$

Set: $W = W(L, \ell, \mathcal{U}) := \frac{L + 1 - w(\mathcal{U})}{\ell + 1}$. Then

$$f(\mathbf{u}) \leq q^{\lfloor W \rfloor} M(f, \mathcal{B}(\mathbf{u}, L + 1)) \leq q^W M(f, \mathcal{B}(\mathbf{u}, L + 1)). \quad (\text{B.5})$$

The proof of Lemma B.1 repeats *verbatim* that of Theorem 2.6.1 in [17], and we omit from the paper.

Lemma B.2 (Cf. Theorem 2.6.2 in [17]). *Fix $0 < \beta, \delta \leq 1$, $m > 0$ and $E \in \mathbb{R}$. Suppose that, for some integer $L \geq 1$ and $\mathbf{u} \in \mathbb{Z}^N$, the N -particle ball $\mathcal{B}(L, \mathbf{u})$ is (E, β) -CNR. Next, take a finite $\mathcal{V} \subset \mathbb{Z}^N$ such that $\mathcal{V} \supset \mathcal{B}(\mathbf{u}, L)$ and $\mathbf{y} \in \mathcal{V} \setminus \mathcal{B}(\mathbf{u}, L)$, and consider the function*

$$f = f_{\mathbf{y}, \mathcal{V}} : \mathbf{x} \in \mathcal{B}(L, \mathbf{u}) \mapsto \left| G_{\mathcal{V}}^{(N)}(\mathbf{x}, \mathbf{y}; E) \right|. \quad (\text{B.6})$$

Given $\ell = 0, \dots, L-1$, let $\Xi = \Xi(E) \subset \mathcal{B}(\mathbf{u}, L - \ell - 1)$ be a (possibly empty) subset such that any ball $\mathcal{B}(\mathbf{x}, \ell) \subset \mathcal{B}(\mathbf{u}, L - \ell - 1) \setminus \Xi$ is (E, δ, m) -NS.

If

$$m\ell^\delta > 2L^\beta > L^\beta + \ln(C_{\mathbb{Z}} L^D) \quad (\text{B.7})$$

then $\forall \mathbf{y} \in \partial^- \mathcal{B}(\mathbf{u}, L)$, function f is (ℓ, q, Ξ) -dominated in $\mathcal{B}(\mathbf{u}, L)$, with

$$q = e^{-m'\ell^\delta}, \quad \text{where } m' := m - 2\ell^{-\delta} L^\beta. \quad (\text{B.8})$$

Proof. First note that for any $\mathbf{x} \in \mathcal{B}(\mathbf{u}, L - \ell) \setminus \Xi$ we have

$$f(\mathbf{x}) \leq e^{-m\ell^\delta} M(f, \mathcal{B}(\mathbf{x}, \ell)),$$

since ball $\mathcal{B}(\mathbf{x}, \ell)$ must be (E, δ, m) -NS, by definition of set Ξ . Clearly, $e^{-m\ell^\delta} < q$, where q is given by (B.8).

Further, define the function $\mathbf{x} \mapsto R_f(\mathbf{x})$ in the same way as in (B.3). Suppose that $\mathbf{x} \in \Xi$ and $R_f(\mathbf{x}) < \infty$, i.e., the spherical layer $\mathcal{L}_{R_f(\mathbf{x})}(\mathbf{u})$ is regular, i.e., each point $\mathbf{y} \in \mathcal{L}_{R_f(\mathbf{x})}(\mathbf{u})$ is regular. Set for brevity $r^* = R_f(\mathbf{x})$. Applying the GRI (2.1) to the ball $\mathcal{B}(\mathbf{u}, r^* - 1)$, we get

$$\begin{aligned} f(\mathbf{x}) &\leq C_{\mathbb{Z}}(r^*)^D \|\mathbf{G}_{\mathcal{B}(\mathbf{u}, r^*-1)}(E)\| \cdot \max_{\mathbf{z} \in \mathcal{L}_{r^*}(\mathbf{u})} |G_{\mathcal{B}(\mathbf{u}, r^*)}(\mathbf{z}, \mathbf{y}; E)| \\ &\leq C_{\mathbb{Z}} L^D e^{L^\beta} M(f, \mathcal{L}_r(\mathbf{u})). \end{aligned}$$

Next, applying the GRI to each ball $\mathcal{B}(\mathbf{z}, \ell)$ with $\mathbf{z} \in \mathcal{L}_r(\mathbf{u})$, we obtain

$$f(\mathbf{x}) \leq C_{\mathbb{Z}} L^D e^{L^\beta} e^{-m\ell^\delta} M(f, \mathcal{L}_{r+\ell}(\mathbf{u})) \leq e^{-m'\ell^\delta} M(f, \mathcal{L}_{r+\ell}(\mathbf{u})),$$

with m' given by (B.8), provided that the condition (B.7) is fulfilled. Thus f is indeed (ℓ, q, Ξ) -dominated in $\mathcal{B}(\mathbf{u}, L)$, with q given by (B.8). \square

Lemma B.2 is used in the proof of Lemmas 3.1 and 4.1.

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